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ON SOME RAMSEY-LIKE STATEMENTS

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“Sono diventato tutto ciò che odiavo e ti assicuro non mi piace”

ABSTRACT

This thesis investigates combinatorial principles from order theory and Ramsey theory with a focus on foundational aspects within the framework of reverse mathematics and computability theory. The first part focuses on *order dimension theory*, a classical topic at the intersection of order theory and combinatorics. Intuitively, the dimension measures how "far" a poset is from being linearly ordered. We are interested in statements that give an upper bound to the dimension of a poset in terms of the dimension of its subposets, obtained by removing one or more points. We analyze these bounding theorems, calibrating their logical strength within subsystems of second order arithmetic. The second part examines the notion of *strong indivisibility*, a particular Ramsey-like property. We focus on countable structures, with particular emphasis on Cameron's classification theorem of strongly indivisible graphs. We study this classification from the perspectives of reverse mathematics and computable combinatorics, investigating the role of induction axioms and effective constructions. In the final part we study a very general *finite Ramsey theorem*, where both the sets being colored and the homogeneous set must satisfy some largeness notion. Historically, largeness notions were associated to countable ordinals and systems of fundamental sequences. To extend this approach we develop a more flexible framework using blocks and barriers. Since the complexity of barriers can be measured by countable ordinals, we define and study Ramsey ordinals, a generalization of the well known Ramsey numbers of classical finite Ramsey theory.

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INTRODUCTION

In this thesis, we apply methods from mathematical logic (particularly those from computability theory) to address a range of problems in combinatorics. Our investigation centers on three main themes: the theory of order dimension, indivisible relational structures and Ramsey-like principles. These areas, while rooted in combinatorial reasoning, also open the door to foundational questions, making them natural subjects for analysis within the frameworks of reverse mathematics and computability theory. We aim to study the computational and logical strength of various combinatorial statements.

Order dimension theory explores the complexity of partially ordered sets by quantifying how far a given poset is from being a linear order. Intuitively, the dimension of a poset measures the minimum number of total orders whose intersection recovers the original partial order. This concept, introduced by Dushnik and Miller, has since become a fundamental tool in understanding the combinatorial and structural properties of partially ordered sets.

Classical results in the theory provide upper bounds on the dimension in terms of parameters like the cardinality, height and width of the partially ordered set, with notable milestones including Hiraguchi's bound stating that an n-element poset has dimension at most n. However, these bounds are often not tight, motivating a deeper investigation into the nature of these inequalities and their logical strength.

This thesis revisits some of these bounding theorems from the point of view of reverse mathematics, a program that seeks to calibrate the precise axiomatic strength required to prove mathematical statements. In Chapter 2 we study which comprehension or induction axioms are necessary for these statements and we classify them within subsystems of second order arithmetic.

Ramsey theory asserts the existence of large homogeneous structures with respect to arbitrary colorings with finitely many colors. While classical Ramsey's theorem and its variants have been intensively studied, recent years have seen a surge of interest in understanding their uniform computational content and the precise logical strength of related combinatorial principles.

One main point of this thesis is the study of indivisible structures. These are countable relational structures that, under any finite coloring of their domain, contain monochromatic induced substructures isomorphic to themselves. We focus on strongly indivisible structures, in which the full set of some color is required to yield an isomorphic substructure. Cameron's classification theorem identifies exactly three strongly indivisible countable graphs: the complete graph, the totally disconnected graph, and the random graph.

In Chapter 3 we examine Cameron's theorem from the perspective of computability theory and reverse mathematics, investigating the complexity of witnessing the indivisibility property. Even though indivisibility and strong indivisibility are similar notions, the latter one turns out to be more difficult to investigate.

The foundational work of Paris and Harrington revealed that certain true natural combinatorial statements cannot be proved in Peano arithmetic. Building on this, subsequent research introduced notions of α-largeness for countable ordinals below ε_0, providing a fine hierarchy to measure the "size" of finite sets of natural numbers.

In this thesis we investigate and develop a more flexible framework of largeness notions using blocks and barriers. This notion generalizes earlier concepts defined via systems of fundamental sequences on ordinals. Each block is assigned a countable ordinal, its height, that captures its combinatorial complexity, enabling the study of largeness beyond classical finite parameters to transfinite settings.

We employ this framework to analyze Ramsey-like statements for colorings of subsets with transfinite size parameters, bridging finite combinatorics and infinite ordinal structures. The central achievement of Chapter 4 is the computation of Ramsey ordinals within this setting, expressed through ordinal functions in the Veblen hierarchy.

While the three parts of this thesis each tackle distinct areas, their common theme is the study of combinatorial principles mostly from Ramsey theory and of foundational questions especially from the perspective of reverse mathematics. Together, these investigations deepen our understanding of how combinatorial principles reflect the landscape of logical strength and computability. This thesis contributes to the broader project of situating combinatorial mathematics within the framework of mathematical logic.

The following chapters develop these themes in detail, providing both classical background and novel results that advance the state of the art in dimension order theory, computable combinatorics and Ramsey theory with large ordinals.

1 PRELIMINARIES

The goal of this first chapter is to give the necessary background and to fix the notation for the rest of the thesis. We do not aim to give an exhaustive presentation of every topic but we provide references in each section. We assume the reader to be familiar with the basic notions of computability theory (as presented, for instance, in [Soa87]).

1.1 REVERSE MATHEMATICS

Reverse mathematics is a research program in the foundations of mathematics started in the nineteen seventies by Friedman and greatly expanded by Simpson and others. Classical references on the topic are [Fri75; Sim09] while a more recent and updated one is [DM22]. Its goal is to calibrate the strength of principles of ordinary mathematics by studying which axioms are necessary (as opposed to sufficient) for proving them. Here by ordinary mathematics we mean mathematics "prior to or independent of abstract set theoretic concepts".

The core idea of reverse mathematics is well summarized by a remarkable observation of Friedman: *"When the theorem is proved from the right axioms, the axioms can be proved from the theorem"*. This phenomenon appears quite often: once a theorem is proved from an appropriate set of axioms, it is possible to derive (over a weak base theory) the axioms from the theorem. This kind of proof is usually called a reversal and gives information about the strength of a theorem. In particular, if a principle T requires a set of axioms S to be proved, then any proof of T must use axioms at least as strong as S.

Now that we understood the aim of reverse mathematics, we need to find a suitable environment to study the mathematical principles. Most of the times the research is developed in *subsystems of second order arithmetic*. The language of second order arithmetic $\mathcal{L}_2$ is a two sorted language, which means that the are two kinds of variables: first order variables (number variables) ranging over $\mathbb{N}$ and second order variables (set variables) ranging over subsets of $\mathbb{N}$. In formulas we use lowercase letters to denote number variables and uppercase letters to denote set variables. The extralogical symbols of $\mathcal{L}_2$ are the following:

- constant symbols $0, 1$,
- binary function symbols $+, \cdot$,
- binary relation symbols $<$ between numbers and $\in$ between a number and a set.

Definition 1.1.1. A model M of $\mathcal{L}_2$ is a tuple

$$(|M|, \mathcal{S}_M, 0_M, 1_M, +_M, \cdot_M, <_M, \in_M)$$

where $|M|$ (the first order part) is the range of number variables, $\mathcal{S}_M \subseteq P(|M|)$ (the second order part) is the range of set variables, 0_M and 1_M are distinguished elements of $|M|$, $+_M$ and $\cdot_M$ and functions from $|M| \times |M|$ to $|M|$ and $<_M$ is a binary relation on $|M|$. The $\in_M$ relation is interpreted as the usual set membership relation.

When working in $\mathcal{L}_2$, it is customary to denote the first-order part of a model by $\mathbb{N}$, and to reserve ω for the standard natural numbers. We also use this convention when working in reverse mathematics.

The *intended model* of $\mathcal{L}_2$ has ω as its first order part and $P(\omega)$ as its second order part, while the extralogical symbols are interpreted in the standard way. An ω-model is a model of $\mathcal{L}_2$ with first order part ω and second order part a Turing ideal, namely a subset of $P(\omega)$ closed under Turing reducibility and join.

Definition 1.1.2. The *arithmetical hierarchy* is the following classification of $\mathcal{L}_2$-formulas without set quantifiers (possibly with number and set parameters). Such formulas are called arithmetical formulas.

- A formula where only bounded number quantifiers occur is Σ^0_0 or Π^0_0.
- If φ is Σ^0_n for some n, then the formula $\forall x\, \varphi$ is Π^0_{n+1}.
- If φ is Π^0_n for some n, then the formula $\exists x\, \varphi$ is Σ^0_{n+1}.
- If φ is equivalent both to a Σ^0_n and to a Π^0_n formula for some n, then φ is Δ^0_n.

Definition 1.1.3. The *analytical hierarchy* is the following classification of $\mathcal{L}_2$-formulas (possibly with number and set parameters).

- An arithmetical formula is Σ^1_0 or Π^1_0.
- If φ is Σ^1_n for some n, then the formula $\forall X\, \varphi$ is Π^1_{n+1}.
- If φ is Π^1_n for some n, then the formula $\exists X\, \varphi$ is Σ^1_{n+1}.

- If φ is equivalent both to a Σ^1_n and to a Π^1_n formula for some n, then φ is Δ^1_n.

Definition 1.1.4. For each family Γ of $\mathcal{L}_2$-formulas, $\mathsf{I}\Gamma$ is the *induction scheme of axioms*

$$(\varphi(0) \land \forall x\, (\varphi(x) \to \varphi(x+1))) \to \forall x\, (\varphi(x))$$

for $\varphi(x) \in \Gamma$. Furthermore, Γ-CA is the *comprehension scheme of axioms*

$$\exists X \forall x\, (x \in X \leftrightarrow \varphi(x))$$

where X is not free in $\varphi(x)$ and $\varphi(x) \in \Gamma$.

The theory Z_2 of *full second order arithmetic* consists of first order axioms of a discrete ordered semiring with a minimum element (a first order theory denoted by PA^-), full induction and full comprehension namely $\mathsf{I}\Gamma$ and Γ-CA where Γ coincides with the set of all $\mathcal{L}_2$-formulas. Full second order arithmetic is an amazingly strong theory and it is rather challenging to find a mathematical theorem that can be expressed as an $\mathcal{L}_2$ formula but cannot be proved in Z_2. Of course, because of Gödel's incompleteness theorems, there are examples like $\mathsf{Con}(\mathsf{Z}_2)$. Other more natural results provable in ZFC (Zermelo–Fraenkel set theory with the axiom of choice) but not in Z_2 are known to set theorists but they are quite sophisticated. An example of this kind of results are choice principles. Z_2 includes full comprehension but is unable to prove choice for Σ^1_3 formulas (see [Sim09, Remark VII.6.3]).

Since Z_2 is so strong and we are interested in calibrating the strength of mathematical theorems using axiomatic systems, we consider fragments obtained by weakening the collection of comprehension and induction axioms. These subsystems are a key subject of reverse mathematics. A typical such subsystem usually has the shape

$$\mathsf{PA}^- + \Gamma\text{-}\mathsf{CA} + \mathsf{I}\Sigma^0_1$$

for some set Γ of $\mathcal{L}_2$ formulas.

A well known empirical fact in reverse mathematics is the so called *big five phenomenon*: many theorems of ordinary mathematics happen to be equivalent to one among four subsystems of second order arithmetic or provable in the base system. We list them in order of strength.

(1) RCA_0: PA^- + Δ^0_1-CA + $\mathsf{I}\Sigma^0_1$,

(2) WKL_0: RCA_0 + Weak König's Lemma,

(3) ACA_0: RCA_0 + comprehension for arithmetical formulas,

(4) ATR_0: RCA_0 + arithmetical transfinite recursion,

(5) Π^1_1-CA: RCA_0 + comprehension for Π^1_1 formulas.

A nice reference for the history of the big five is [DW17].

RCA_0 corresponds roughly speaking to a formalization of *computable mathematics*. Essentially RCA_0 guarantees that two principles are equivalent up to effective transformation. Usually it is assumed to be the weak base theory over which the reversals are proved. Despite its weakness, RCA_0 is sufficient to prove a number of classical theorems like the intermediate value theorem or Szpilrajn theorem. These theorems are, in a sense, below the reach of the reverse mathematics enterprise because they are already provable in the base system.

WKL_0 is the strengthening of RCA_0 with the statement "every infinite binary tree has an infinite path". It corresponds to *finitistic reductionism* (see [Sim09, Section I.12]). Such system is strictly stronger than RCA_0 and there are many mathematical principles equivalent to it. We mention the ones most pertinent to our purposes.

Theorem 1.1.5 (RCA_0)**.** *The following are equivalent:*

(1) WKL_0*,*

(2) if $f, g\colon \mathbb{N} \to \mathbb{N}$ *are 1-1 functions with* $\forall m \forall n\,(f(m) \neq g(n))$*, then* $\exists X \forall n\,(f(n) \in X \land g(n) \notin X)$*,*

(3) if $\varphi(n)$ *and* $\psi(n)$ *are* Σ^0_1 *formulas in which* X *is not free and* $\neg\exists n\,(\varphi(n) \land \psi(n))$*, then* $\exists X \forall n\,(\varphi(n) \to n \in X \land \psi(n) \to n \notin X)$*.*

ACA_0 corresponds to *predicativism*. The basic ideas of this axiomatic system were already present in the work of Russell and Poincaré at the beginning of the twentieth century. The system is a conservative extension of first order Peano arithmetic. We point out a couple of useful equivalent mathematical principles.

Theorem 1.1.6 (RCA_0)**.** *The following are equivalent:*

(1) ACA_0*,*

(2) if $f\colon \mathbb{N} \to \mathbb{N}$ *is a 1-1 function, then* $\exists X \forall n\,(n \in X \leftrightarrow \exists m\,(f(m) = n))$*,*

(3) for each set X *its Turing jump* X' *exists i.e.* $\forall X \exists Y\,(Y = X')$*.*

The remaining systems are less important for the rest of the thesis. ATR_0 corresponds to *predicative reductionism* and it states, informally, that any arithmetical functional can be iterated along any countable well ordering. Π^1_1-CA_0 is *fully impredicative* and in some sense it is related to ATR_0 as ACA_0 is related to WKL_0.

Theorem 1.1.7 (RCA_0)**.** *The following hold:*

- ATR_0 *is equivalent to the statement: for every* Σ^1_1 *formulas* $\varphi(n)$ *and* $\psi(n)$ *in which* X *is not free and* $\neg\exists n\,(\varphi(n) \wedge \psi(n))$, *then* $\exists X \forall n\,(\varphi(n) \to n \in X \wedge \psi(n) \to n \notin X)$,
- Π^1_1-CA_0 *is equivalent to the statement: each ill founded tree has a leftmost path.*

The big five derive their strength from set existence axioms. In fact, they can also be interpreted as statements asserting the existence of more and more incomputable sets, linking them to fundamental results in computability theory. By contrast, alternative subsystems can be obtained by strengthening induction rather than comprehension.

Definition 1.1.8. For each family Γ of $\mathcal{L}_2$-formulas, $\mathsf{B}\Gamma$ is the *bounding scheme of axioms*

$$\forall z\,(\forall x < z \exists y\, \varphi(x,y) \to \exists w \forall x < z \exists y < w\, \varphi(x,y))$$

for $\varphi(x) \in \Gamma$. Furthermore, $\mathsf{L}\Gamma$ is the *least number principle scheme of axioms*

$$\exists x\, \varphi(x) \to \exists x\,(\varphi(x) \wedge \neg\exists y < x\, \varphi(y))$$

for $\varphi(x) \in \Gamma$.

A lot of work concerning $\mathsf{I}\Gamma$, $\mathsf{B}\Gamma$ and $\mathsf{L}\Gamma$ was done by Paris and Kirby in [PK78]. We summarize the most important results.

Theorem 1.1.9. *Fix* $n \geqslant 1$. *The following are provable in* PA^-*:*

- $\mathsf{B}\Sigma^0_n$ *is equivalent to* $\mathsf{B}\Pi^0_{n-1}$,
- $\mathsf{I}\Sigma^0_{n-1} + \mathsf{B}\Sigma^0_0$ *implies* $\mathsf{B}\Sigma^0_{n-1}$,
- $\mathsf{I}\Sigma^0_0 + \mathsf{B}\Sigma^0_n$ *implies* $\mathsf{I}\Sigma^0_{n-1}$.

Moreover, the following equivalences hold $\mathsf{I}\Sigma^0_n \leftrightarrow \mathsf{I}\Pi^0_n \leftrightarrow \mathsf{L}\Sigma^0_n \leftrightarrow \mathsf{L}\Pi^0_n$.

It is provable that the implications in the previous theorem are strict. Hence, we obtain a strictly increasing hierarchy of subsystems of Z_2 which is known as the Paris-Kirby hierarchy. If we add RCA_0 to each subsystem, we have that the hierarchy lies between RCA_0 and ACA_0 in terms of axiomatic strength. At the same time, each principle is incomparable with WKL_0.

We are mostly interested in $\mathsf{I}\Sigma^0_2$ in Chapter 2 and Chapter 3. Theorems equivalent to this system have been studied for instance in [Hir92; GHM15; CT22; Fri17]. We recall here a basic fact (see [Simo9, Exercise II.3.13]).

Theorem 1.1.10 (RCA_0). *The following are equivalent:*

(1) $\mathsf{I}\Sigma^0_2$,

(2) bounded Σ^0_2 *comprehension: for each* Σ^0_2 *formula* φ *with* X *not free*

$$\forall z \exists X \forall x \, (x \in X \leftrightarrow x < z \wedge \varphi(x)).$$

We highlight that there are theorems of ordinary mathematics which are not equivalent to any of the big five axiom systems. Starting from Seetapun's groundbreaking result that RT^2_2 (Ramsey theorem for pairs and two colors) is not equivalent to any of the big five, many statements mostly from combinatorics and Ramsey theory that lie strictly between RCA_0 and ACA_0 and are incomparable with WKL_0 have been discovered. This phenomenon became known as the *zoo of reverse mathematics*.

1.2 GRAPH THEORY AND ORDER THEORY

For a deeper introduction to the topics of this section we refer to [Fis85; Har05].

Formally, a countable *graph* $G = (V, E)$ is a pair consisting of a nonempty set $V \subseteq \mathbb{N}$ of vertices and an irreflexive, symmetric edge relation E. For $W \subseteq V$, the *induced subgraph* is $(W, E \upharpoonright W \times W)$. We frequently abuse notation by equating a graph with its domain (e.g. writing $W \subseteq G$), by conflating a set of vertices with its induced subgraph, and by using subgraph to mean induced subgraph.

A vertex x is *isolated* if $\neg E(x,y)$ for all $y \in G$ and is *universal* if $E(x,y)$ for all $y \neq x$ in G. A countable graph $G = (V, E)$ is *strongly indivisible* if for every vertex partition $V = W_0 \sqcup W_1$, either $W_0 \cong G$ or $W_1 \cong G$.

For $n > 0$, K_n denotes the complete graph on n vertices. K_ω denotes the complete graph with $V = \mathbb{N}$ and $E = \{\langle n, m \rangle : m \neq n\}$, and $\overline{K}_\omega$ denotes the completely disconnected graph with $V = \mathbb{N}$ and $E = \emptyset$.

For a graph $G = (V, E)$, let $\overline{G}$ denote the graph obtained by swapping the edges and non edges (except along the diagonal). Formally, $\overline{G} = (V, \overline{E})$ with $\overline{E} = \{\langle m, n \rangle \notin E : m \neq n\}$.

The standard development of the random graph can be carried out in RCA_0. Let $\varphi_{\mathcal{R}}$ denote the usual extension axiom stated in second order arithmetic.

$$\forall \text{finite}\, A, B \subseteq V \Big(A \cap B = \emptyset \to \exists x \, ((\forall a \in A) \, E(x, a) \wedge (\forall b \in B) \neg E(x, b)) \Big).$$

A countable graph satisfying $\varphi_{\mathcal{R}}$ is called a *random graph*. When working in a random graph, we can assume the existential witness x satisfies $x \notin A \cup B$. To see why, apply $\varphi_{\mathcal{R}}$ to $A \cup B$ and $\emptyset$ to get a vertex v connected to every node in $A \cup B$. Then apply $\varphi_{\mathcal{R}}$ to A and $B \cup \{v\}$ to get a node x connected to everything in A and nothing in $B \cup \{v\}$. It follows that $x \notin A$ because the edge relation is irreflexive and $x \notin B$ because everything in B is connected to v.

RCA_0 suffices to show that there is a random graph by, for example, letting $V = \mathbb{N}$ and putting a symmetric edge between x and y when $x < y$ and the x-th bit of the binary representation of y is 1. Moreover, the classical back and forth argument can be carried out in RCA_0 to show that there is a unique random graph up to isomorphism.

Proposition 1.2.1 (RCA_0)**.** *If* G_0 *and* G_1 *are random graphs, then* $G_0 \cong G_1$.

We continue to use $\mathcal{R}$ to denote a random graph, which by Proposition 1.2.1, is determined up to isomorphism in RCA_0.

A *partially ordered set* (or simply a *poset*) is a pair $(P, \preceq)$ where P is a set called the domain and $\preceq$ is a binary relation which is reflexive, transitive and antisymmetric. As in the graph case, we often abuse notation by identifying a poset with its domain. We denote by $\prec$ the irreflexive version of $\preceq$. We introduce another binary relation $|$ associated to a poset $(P, \preceq)$. We stipulate that $x \mid y$ if and only if $x \npreceq y$ and $y \npreceq x$. We call $|$ the *incomparability relation* of $(P, \preceq)$. We say that subsets Y and Z of a poset $(P, \preceq)$ are incomparable if for each $y \in Y$ and each $z \in Z$, $y \mid z$.

A subset X of a poset $(P, \preceq)$ is *downward closed* if for every $x \in X$ and every $y \in P$, if $y \preceq x$ then $y \in X$. We also say that X is an *initial interval* of $(P, \preceq)$.

A *linear order* (or a *chain*) is a poset $(P, \trianglelefteq)$ in which any two distinct elements are comparable. We use the symbol $\preceq$ to denote a generic poset and we reserve the symbol $\trianglelefteq$ if we want to highlight that it is a linear order. We say that a poset $(P, \preceq_1)$ extends a poset $(P, \preceq_2)$ if $\preceq_2 \subseteq \preceq_1$. An extension of a poset to a linear order is called a *linearization*.

Theorem 1.2.2 (Szpilrajn extension theorem)**.** *Every poset can be linearized.*

The classical proof can be found in [Szp30] and uses Zorn's lemma. Theorem 1.2.2 implies the axiom of finite choice (see [Her06]) which states that if $(S_\alpha)_{\alpha \in I}$ is a family of non empty finite sets then the set theoretic product $\Pi_{\alpha \in A} S_\alpha$ is non empty. This statement is strictly weaker than full axiom of choice but it is still independent from ZF (see [Moo82]). In [HR98] it is shown that Theorem 1.2.2 combined with the statement that every total order has a cofinal well order, proves the full axiom of choice.

It is possible to define a number of parameters to describe a partial order. For instance the *height* measures the maximum cardinality of a chain while the *width* measures the maximum cardinality of an antichain. Among these parameters one of the most interesting is the *dimension* of a poset.

Definition 1.2.3. If $(P, \preceq)$ is a poset, we say that a family $(P, \trianglelefteq_i)_{i \in I}$ of linearizations realize $(P, \preceq)$ if $\bigcap_{i \in I} \trianglelefteq_i = \preceq$. The dimension of $(P, \preceq)$ is the least cardinality of a realization and is denoted $\dim(P, \preceq)$ or simply by $\dim(P)$.

By Theorem 1.2.2 there exists at least a linearization of any poset. A poset has dimension 1 if and only if it is a chain. On the other hand, an antichain has dimension 2: it suffices to take any linearization $\trianglelefteq$ of the antichain and its reverse $\trianglerighteq$. A fundamental result proved by Dushnik and Miller in [DM41] states that for any cardinal $\kappa > 0$ there exists a poset of dimension κ.

In practice, to show that $\dim(P) \leqslant \kappa$ it suffices to find a set of κ linearizations $\{\trianglelefteq_\alpha : \alpha < \kappa\}$ of P satisfying: for all $x, y \in P$ such that $x \mid y$ there exists $\alpha < \kappa$ with $x \ntrianglelefteq_\alpha y$. In fact, if $x \preceq y$ then $x \trianglelefteq_\alpha y$ holds for every α, and if $y \preceq x$ then $x \trianglelefteq_\alpha y$ never holds.

1.3 SYSTEMS OF FUNDAMENTAL SEQUENCES

If $X \subseteq \mathbb{N}$ we denote by $[X]^{<\omega}$ the set of finite subsets of X and, for $n \in \mathbb{N}$, by $[X]^n$ the set of subsets of X with exactly n elements. In this section and in the one that follows we use uppercase letters X, Y, Z to denote subsets of $\mathbb{N}$ which may be finite or infinite and we use lowercase letters s, t, u to denote finite subsets of $\mathbb{N}$.

We identify a subset of $\mathbb{N}$ with the strictly increasing sequence (finite or infinite) which enumerates it and denote by $X(i)$ or X_i the element in position i in the sequence. If s is such a finite sequence then $|s|$ denotes its length (coinciding with the cardinality of the set) and we write $s = \langle s_0, \dots, s_{|s|-1} \rangle$.

Given $s, X \subseteq \mathbb{N}$, by $s \sqsubseteq X$ we mean that s, as a sequence, is an initial segment (or prefix) of X. This is stronger than $s \subseteq X$, which denotes set theoretic inclusion as usual. The irreflexive versions of the previous relations are denoted by $\sqsubset$ and $\subset$.

If s and X are non empty, we let $s^* = s \setminus \{\max s\}$ and $X^- = X \setminus \{\min X\}$. By $s < X$ we mean that each element of s is strictly smaller than each element of X or equivalently (when s and X are nonempty) $\max s < \min X$. We write $s^\frown X$ for the concatenation of the sequences s and X. Notice that in our setting $s^\frown X$ does make sense only if $s < X$ and, set theoretically, coincides with $s \cup X$.

For ordinals α and β we write $\alpha \ggg \beta$ to mean that the exponent of the last term in the Cantor normal form of α is greater than or equal to the exponent of the first term in the Cantor normal form of β.

Definition 1.3.1. A *fundamental sequence* for an ordinal $\alpha > 0$ is a non decreasing sequence of ordinals $\alpha[n]$ such that $\sup\{\alpha[n]+1 : n \in \omega\} = \alpha$. For notational convenience we stipulate that the fundamental sequence of 0 is always $0[n] = 0$ for all n.

Let Γ be an ordinal. If we fix a fundamental sequence for each ordinal $\alpha < \Gamma$ we speak of a *system of fundamental sequences on* Γ.

Definition 1.3.2. A system of fundamental sequences on Γ is *regular* if for each ordinal $\beta_0 + \omega^{\beta_1} < \Gamma$ with $\beta_0 \gg \omega^{\beta_1}$ (i.e. β_0 are the leading terms in the Cantor normal form of the ordinal, which has another term) and each n it holds that

$$(\beta_0 + \omega^{\beta_1})[n] = \beta_0 + (\omega^{\beta_1}[n]).$$

Notice that if α is an ordinal such that $\omega^{\beta} < \alpha < \omega^{\beta+1}$, regularity implies that $\alpha[n] \geqslant \omega^{\beta}$ for each n. Regularity also implies that for each α and each n, $(\alpha+1)[n] = \alpha$.

Definition 1.3.3. A system of fundamental sequences on Γ is *nested* if it is never the case for $\gamma < \beta < \Gamma$ and $n > 1$ that

$$\gamma > \beta[n] > \gamma[n].$$

Regularity is a very natural property that is satisfied by most systems of fundamental sequences that have been studied. Nestedness is a weakening of the Bachmann property which states that for ordinals $\gamma < \beta < \Gamma$ it is never the case that $\gamma > \beta[n] > \gamma[1]$. The Bachmann property was introduced in [Bac67] and studied in [PTV91]. An example of a system of fundamental sequences that satisfies the Bachmann property (but, incidentally, is not regular) is provided in [FW24].

Various systems of fundamental sequences have been developed, and they are all quite similar. They are usually developed on an ordinal for which we have some ordinal notation. In Chapter 4 instead, we define a way of going from a system of fundamental sequences on an ordinal ζ to a system of fundamental sequences on the ordinal Γ_ζ – the ζ-th fixed point for the binary Veblen function. Recall that the unary Veblen functions are a hierarchy of normal functions introduced by Veblen in [Veb08]. The function φ_0 is ordinal exponentiation in base ω and for each ordinal $\alpha > 0$, φ_α enumerates the common fixed points of φ_β for $\beta < \alpha$ in strictly increasing order. Since the Veblen functions are normal, each of them has infinitely many fixed points. Then one can define the binary Veblen function $\varphi(\alpha, \beta) = \varphi_\alpha(\beta)$: its fixed points are the ordinals α such that $\alpha = \varphi(\alpha, 0)$. Hence, for each ζ, Γ_ζ is the ζ-th ordinal α such that $\varphi_\alpha(0) = \alpha$.

1.4 FRONTS, BLOCKS AND BARRIERS

Blocks and barriers are combinatorial objects introduced in [Nas65] to define better quasi orderings, a refinement of the notion of well quasi orderings. Laver later used better quasi orderings in [Lav71] to prove Fraïssé's order type conjecture. Here we give the definition of a front (see [CP20]), which is a slightly more general object than a block. We consider also smooth barriers, which were introduced in [Mar94].

Definition 1.4.1. A set $\mathcal{B} \subseteq [\mathbb{N}]^{<\omega}$ is a *front* if either $\mathcal{B} = \{\langle\rangle\}$ or if it satisfies the following properties:

(1) $\mathrm{base}(\mathcal{B}) = \{n \in \mathbb{N} : \exists s \in \mathcal{B}\, \exists i < |s|\,(s(i) = n)\}$ is infinite,

(2) for each infinite $X \subseteq \mathrm{base}(\mathcal{B})$ there is some $s \in \mathcal{B}$ such that $s \sqsubset X$,

(3) for each $s, t \in \mathcal{B}$ $s \not\sqsubset t$.

We say that $\mathcal{B} = \{\langle\rangle\}$ is the *degenerate front* and by convention we say that $\mathrm{base}(\{\langle\rangle\}) = \mathbb{N}$. A *block* is a non degenerate front.

A *barrier* is a front $\mathcal{B}$ such that for each $s, t \in \mathcal{B}$ $s \not\subset t$.

A front $\mathcal{B}$ is *smooth* if for each $s, t \in \mathcal{B}$ with $|s| < |t|$ there exists $i < |s|$ with $s(i) < t(i)$.

For every n, $\{s \in [\mathbb{N}]^{<\omega} : |s| = n\}$ is a smooth barrier. For convenience, we denote by $\mathbb{1}$ and $\mathbb{2}$ the barriers of singletons and pairs.

The set $\{s \in [\mathbb{N}]^{<\omega} : |s| = s(0) + 1\}$ is also a smooth barrier, often called the Schreier barrier.

A subfront (respectively, a subblock, a subbarrier) is a subset of a front which is still a front (respectively, a block, a barrier). If $\mathcal{B}$ is a block and $\mathcal{B}' \subseteq \mathcal{B}$ is a subblock of $\mathcal{B}$ then it must be $\mathcal{B}' = \{s \in \mathcal{B} : s \subseteq X\}$ for some infinite $X \subseteq \mathrm{base}(\mathcal{B})$. Vice versa, for each infinite $X \subseteq \mathrm{base}(\mathcal{B})$ we obtain a subblock of $\mathcal{B}$ by restricting to the elements of $\mathcal{B}$ which are subsets of X. It is straightforward to see that a smooth block is automatically a barrier. It is also immediate that any subblock of a (smooth) barrier is a (smooth) barrier.

Notice that up to isomorphism we may assume that the base of a front is $\mathbb{N}$. We often tacitly do that in proofs where only one block is involved and no restrictions on its base are required.

Every front is well ordered by lexicographic order [Pou72] and we denote by $\mathrm{o.t.}(\mathcal{B})$ its *order type*. The degenerate front is the only front of order type 1. While the order type of blocks can be any countable limit ordinal, it can be proved that the order types of barriers are exactly the ordinals of the form $\omega^\beta \cdot n$ for $0 \leqslant \beta < \omega_1$, $0 < n < \omega$ and if $\beta < \omega$ then $n = 1$ (see [Ass74]). Any barrier of order type $\omega^\beta \cdot n$ contains a subbarrier of order type

ω^β. The order type of each smooth barrier is indecomposable (i.e. of the form ω^β for some $0 \leqslant \beta < \omega_1$) and for each β with $0 \leqslant \beta < \omega_1$ there exists a smooth barrier with order type ω^β (see [Mar94]). However, there exist barriers with indecomposable order type which are not smooth.

Another measure of the complexity of a front is its *height*. Let $\mathcal{B}$ be a front and let

$$\mathsf{T}(\mathcal{B}) = \{s \in [\mathrm{base}(\mathcal{B})]^{<\omega} : \forall t\,(t \sqsubset s \to t \notin \mathcal{B})\}.$$

In other words $\mathsf{T}(\mathcal{B})$ is the closure under initial segments of $\mathcal{B}$. Then $\mathsf{T}(\mathcal{B})$ is a well founded tree and the elements of $\mathcal{B}$ are its leaves.

We recursively define a function that assigns to each node $s \in \mathsf{T}(\mathcal{B})$ a countable ordinal $\mathrm{ht}_{\mathcal{B}}(s)$. If the front is clear from the context we simply write $\mathrm{ht}(s)$. If $s \in \mathcal{B}$ then $\mathrm{ht}_{\mathcal{B}}(s) = 0$ while if $s \in \mathsf{T}(\mathcal{B}) \setminus \mathcal{B}$ then $\mathrm{ht}(s) = \sup\{\mathrm{ht}(t) + 1 : t \text{ is an immediate extension of } s \text{ in } \mathsf{T}(\mathcal{B})\}$. The *height* $\mathrm{ht}(\mathcal{B})$ of the front $\mathcal{B}$ is $\mathrm{ht}_{\mathcal{B}}(\langle\rangle)$. The degenerate front is the only front with height 0. Marcone proved the following in [Mar94].

Lemma 1.4.2. *If $\mathcal{B}$ is a block, then there exists a smooth barrier $\mathcal{B}'$ such that* $\mathrm{base}(\mathcal{B}) = \mathrm{base}(\mathcal{B}')$, $\mathrm{ht}(\mathcal{B}) = \mathrm{ht}(\mathcal{B}')$ *and* $\mathcal{B} \subset \mathsf{T}(\mathcal{B}')$.

Definition 1.4.3. If $\mathcal{A}$ and $\mathcal{B}$ are fronts such that $\mathrm{base}(\mathcal{B}) = \{n \in \mathrm{base}(\mathcal{A}) : n \geqslant m\}$ for some m (we say that $\mathrm{base}(\mathcal{B})$ is a *final segment* of $\mathrm{base}(\mathcal{A})$) then the *sum* $\mathcal{A} \oplus \mathcal{B}$ is the front $\{s^\frown t : s \in \mathcal{B}, t \in \mathcal{A}, \max s < \min t\}$.

If the base of one of the fronts $\mathcal{A}$ and $\mathcal{B}$ is a final segment of the other we define the *union* $\mathcal{A} \sqcup \mathcal{B}$ to be the front consisting of the leaves of $\mathsf{T}(\mathcal{A}) \cup \mathsf{T}(\mathcal{B})$.

If we apply either operation to smooth barriers we obtain a smooth barrier.

Remark 1.4.4. Notice that $\mathrm{base}(\mathcal{A} \oplus \mathcal{B}) = \mathrm{base}(\mathcal{B})$ and $\mathrm{base}(\mathcal{A} \sqcup \mathcal{B}) = \mathrm{base}(\mathcal{A}) \cup \mathrm{base}(\mathcal{B})$.

Arguing by induction and assuming that $\mathcal{A}$ and $\mathcal{B}$ are smooth barriers[1], for each $s \in \mathsf{T}(\mathcal{B})$ $\mathrm{ht}_{\mathcal{A}\oplus\mathcal{B}}(s) = \mathrm{ht}(\mathcal{A}) + \mathrm{ht}_{\mathcal{B}}(s)$ and hence $\mathrm{ht}(\mathcal{A} \oplus \mathcal{B}) = \mathrm{ht}(\mathcal{A}) + \mathrm{ht}(\mathcal{B})$ (this explains the order of the summands in our notation).

Again an easy induction shows that if $\mathcal{A}$ and $\mathcal{B}$ are fronts then for each $s \in \mathsf{T}(\mathcal{A} \sqcup \mathcal{B})$ it holds $\mathrm{ht}_{\mathcal{A}\sqcup\mathcal{B}}(s) = \max(\mathrm{ht}_{\mathcal{A}}(s), \mathrm{ht}_{\mathcal{B}}(s))$, where we stipulate that $\mathrm{ht}_{\mathcal{A}}(s) = -1$ if $s \notin \mathsf{T}(\mathcal{A})$ (and the same for $\mathcal{B}$). It follows that $\mathrm{ht}(\mathcal{A} \sqcup \mathcal{B}) = \max(\mathrm{ht}(\mathcal{A}), \mathrm{ht}(\mathcal{B}))$.

If $\mathcal{B}$ is a block and $s \in \mathsf{T}(\mathcal{B})$ then we define the tree $\mathsf{T}_s(\mathcal{B}) = \{t : s^\frown t \in \mathsf{T}(\mathcal{B})\}$ and we call $\mathcal{B}_s$ the set of its leaves, so that $\mathcal{B}_s = \{t \in [\mathrm{base}(\mathcal{B})]^{<\omega} : s^\frown t \in \mathcal{B}\}$. One can check that $\mathcal{B}_s$ is a front, $\mathrm{ht}_{\mathcal{B}}(s) = \mathrm{ht}(\mathcal{B}_s)$ and if $\mathcal{B}_s$ is non degenerate then $\mathrm{base}(\mathcal{B}_s) = \mathrm{base}(\mathcal{B}) \setminus \{0, \dots, \max s\}$. Moreover, if $\mathcal{B}$ is a smooth barrier then $\mathcal{B}_s$ is a smooth barrier too. If $s = \langle n \rangle$ then we simply write $\mathsf{T}_n(\mathcal{B})$ and $\mathcal{B}_n$ instead of $\mathsf{T}_{\langle n \rangle}(\mathcal{B})$ and $\mathcal{B}_{\langle n \rangle}$.

[1] this might fail when $\mathcal{A}$ is not a smooth barrier.

2 DIMENSION OF POSETS

This chapter is joint work with Marta Fiori Carones and Alberto Marcone.

Order dimension theory lies at the intersection of *order theory* and *combinatorics* and it provides a framework for understanding the complexity of *partially ordered sets* (posets). Informally, the dimension measures how "far" a poset is from being totally ordered. The greater the number of linear extensions required to describe a poset, the more intricate its structure is.

The study of poset dimension began with the seminal work of Dushnik and Miller in [DM41], where the concept was introduced as a way to represent partial orders as intersections of linear orders. After that, research in the area was pursued by Trotter, Hiraguchi and many others who explored combinatorial and structural properties of poset dimension. The field expanded rapidly through the 1970s and 1980s (see e.g. [BFR72; BJ73; Bog73; Jr75; Rab78a; Rab78b]). The monograph by Trotter [Tro92] became a central reference in the area, collecting and organizing many results.

Dimension, intuitively, measures the complexity of the ordering. One might expect that a poset with low dimension is close to being linearly ordered, but this expectation is not always accurate: there are posets whose dimension is small but whose structure is rather complicated, as shown in [Tro92].

Several structural parameters are closely related to dimension, such as cardinality, height and width. For example, the dimension of a poset $(P, \preceq)$, is bounded above both by its width (see [Dil50]) and by $\frac{|P|}{2}$ (see [Hir51]). These bounds are tight: the standard example of a poset of dimension n consists of the 1-element and $(n-1)$-element subsets of an n-element set ordered by inclusion and it has $2n$ elements, width n and dimension n. We introduce such poset in Definition 2.1.5 and we call it F_n.

Other significant contributions come from work of Baker, who showed that the dimension of a poset is at least as great as its breadth (a proof can be found in [Fis85]). This provides a useful lower bound and complements the upper bounds by Hiraguchi and others.

An important feature of dimension is its monotonicity: a subposet cannot have greater dimension than the starting poset. In addition, dimension is continuous in the sense that small changes to a poset cannot drastically alter its dimension. These features make the dimension robust under typical constructions and modifications.

A particularly rich area of interaction is the connection between order theory and the *foundations of mathematics* through the framework of *reverse mathematics*. In this context, researchers analyze the logical strength of mathematical theorems by determining the minimal axiomatic systems needed to prove them. Many results in order theory — such as variants of Dilworth's theorem, initial interval separation for posets, WQO and BQO theory — have been studied within subsystems of second-order arithmetic (see e.g. [CMS04; FM14; Mar94]).

In this chapter we aim to study classical bounding results about order dimension theory from the point of view of reverse mathematics.

In Section 2.1 we recall classical results from order theory. We also give basic definitions that we need in the rest of the chapter.

In Section 2.2 we state the bounding theorems we are interested in and we provide examples to show that these bounds are sharp.

In Section 2.3 and 2.4 we give proofs of the bounding theorems (highlighting which subsystems of second order arithmetic are used) and we provide reversals of the statements studied.

2.1 BASIC RESULTS

Theorem 1.2.2 restricted to countable posets is provable in RCA_0. Moreover, the proof is uniform and so we get the following.

Theorem 2.1.1 (RCA_0). *For each sequence of posets* $(P_i, \preceq_i)_{i\in\mathbb{N}}$ *there exists a sequence of linearizations* $(P_i, \trianglelefteq_i)_{i\in\mathbb{N}}$.

Proof. Let $\langle\cdot,\cdot\rangle\colon \mathbb{N}\times\mathbb{N}\to\mathbb{N}$ be a computable bijection and for each $i\in\mathbb{N}$ fix a listing $(x^i_n)_{n\in\mathbb{N}}$ of the elements of P_i. We define simultaneously for each i a linear order $\trianglelefteq_i$ which extends $\preceq_i$ and is $\preceq_i$ computable. We proceed by stages. Suppose we are at stage $s=\langle i,m\rangle$: we want to stipulate the $\trianglelefteq_i$ comparabilities of x^i_m. For each $\langle i,n\rangle<s$, if there is $\langle i,n'\rangle<s$ such that $x^i_{n'}\preceq_i x^i_m$ and $x^i_n \trianglelefteq_i x^i_{n'}$, then we put $x^i_n\trianglelefteq_i x^i_m$. Otherwise we put $x^i_m\trianglelefteq_i x^i_n$. Then for each i each relation $\trianglelefteq_i$ extends the corresponding $\preceq_i$ and for any $x,y\in P_i$ such that $x\neq y$, either $x\trianglelefteq_i y$ of $y\trianglelefteq_i x$. Moreover, $\trianglelefteq_i$ is still a poset and it is computable from $\preceq_i$ as required. $\square$

We can also prove that a realization of a poset (Definition 1.2.3) always exists in RCA_0 by formalizing an argument of [DM41].

Theorem 2.1.2 (RCA_0). *For each poset* $(P, \preceq)$*, there exists a set* $\{\trianglelefteq_n : n \in \mathbb{N}\}$ *of linearizations of* $(P, \preceq)$ *which realizes it.*

Proof. If $(P, \preceq)$ is a chain, then $\{\preceq\}$ is a realization.

Assume that $(P, \preceq)$ is not a chain and let $a, b \in P$ be such that $a \mid b$. We extend $(P, \preceq)$ to another poset $(P, \preceq^b_a)$ in the following way: for each $x, y \in P$ we stipulate that $x \preceq^b_a y$ if and only if either $x \preceq y$ or $x \preceq a \wedge b \preceq y$. Notice that such relation exists in RCA_0, $\preceq \subseteq \preceq^b_a$ and $a \preceq^b_a b$.

We claim that $(P, \preceq^b_a)$ is a poset. Reflexivity and antisymmetry are trivial. To prove transitivity let $x, y, z \in P$ be such that $x \preceq^b_a y$ and $y \preceq^b_a z$ and proceed by cases. If $x \preceq y$ and $y \preceq z$ the conclusion follows from transitivity of $\preceq$. The case $x \preceq a$, $b \preceq y$, $y \preceq a$ and $b \preceq z$ cannot occur because it would imply $b \preceq y \preceq a$, against the assumption $a \mid b$. Suppose that $x \preceq y$, $y \preceq a$ and $b \preceq z$: by transitivity of $\preceq$ we obtain $x \preceq a$ and thus $x \preceq^b_a z$ by definition. The case $x \preceq a$, $b \preceq y$ and $y \preceq z$ is analogous. Reflexivity and antisymmetry are proved again by cases. We conclude that $(P, \preceq^b_a)$ is a poset that extends $(P, \preceq)$.

By Theorem 2.1.1, for each $a, b \in P$ such that $a \mid b$, there exists a linearization $\trianglelefteq^b_a$ of $(P, \preceq^b_a)$. It is clear that if $x \mid y$ then $x \not\trianglelefteq^x_y y$, so that $\{\trianglelefteq^b_a : a, b \in P, a \mid b\}$ realizes $(P, \preceq)$. ☐

Notice that some technical results that are useful in the following are available in WKL_0.

Theorem 2.1.3 (RCA_0). *The following are equivalent:*

(1) WKL_0,

(2) every acyclic relation can be extended to a partial order,

(3) for every partial order $(P, \preceq)$ *and sets* $I, F \subseteq P$ *such that* $\forall x \in I \forall y \in F (y \npreceq x)$ *there exists a downward closed set* $B \subseteq P$ *such that* $I \subseteq B$ *and* $B \cap F = \emptyset$.

The equivalence of WKL_0 with (2) was proved in [CMS04], while the equivalence with (3) was proved in [FM14].

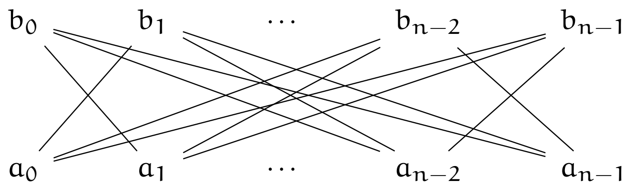
The statement (2) in Theorem 2.1.3 self-strengthens to its sequential version.

Corollary 2.1.4 (WKL_0). *If* $(R_i)_{i \in \mathbb{N}}$ *are acyclic relations on sets* $(P_i)_{i \in \mathbb{N}}$ *respectively, then each* R_i *can be extended to a partial order.*

Proof. Let P and R be the disjoint unions respectively of $(P_i)_{i \in \mathbb{N}}$ and $(R_i)_{i \in \mathbb{N}}$. Then R is an acyclic relation on the set P and we can apply (2) of Theorem 2.1.3 to get a partial order $(P, \preceq)$ which extends R. Each restriction of $\preceq$ to P_i is a partial order extending R_i. ☐

We show that for each $n \in \mathbb{N}$ there exists a poset of dimension exactly n.

Definition 2.1.5. Let $n > 1$ and let $F_n = \{a_i, b_i : i < n\}$. Equip F_n with the partial order $\prec = \{(a_i, b_j) : i \neq j\}$.



For simplicity, we will refer to this poset by F_n.

Theorem 2.1.6 (RCA_0). *The poset* F_n *has dimension* n.

Proof. To prove that $\dim(F_n) \geqslant n$ it suffices to show that for every linearization $\trianglelefteq$ of F_n there exists at most one $k < n$ such that $b_k \trianglelefteq a_k$. In fact, if $b_k \trianglelefteq a_k$ and $b_{k'} \trianglelefteq a_{k'}$ for $k \neq k'$, then

$$b_{k'} \trianglelefteq a_{k'} \trianglelefteq b_k \trianglelefteq a_k \trianglelefteq b_{k'},$$

and by antisymmetry of $\trianglelefteq$ we obtain $a_k = a_{k'} = b_k = b_{k'}$, a contradiction.

For the converse it suffices to define a realization consisting of exactly n linearizations. For each $i < n$ let $\trianglelefteq_i$ be the following linearization of F_n

$$a_0 \trianglelefteq_i \ldots \trianglelefteq_i a_{i-1} \trianglelefteq_i a_{i+1} \trianglelefteq_i \ldots \trianglelefteq_i a_{n-1} \trianglelefteq_i b_i \trianglelefteq_i$$
$$\trianglelefteq_i a_i \trianglelefteq_i b_{n-1} \trianglelefteq_i \ldots b_{i+1} \trianglelefteq_i b_{i-1} \trianglelefteq_i \ldots \trianglelefteq_i b_0.$$

Each $\trianglelefteq_i$ exists in RCA_0 by $\boldsymbol{\Sigma}^0_0$-comprehension. We claim that $\{\trianglelefteq_0, \ldots, \trianglelefteq_{n-1}\}$ realizes F_n. If $x \mid y$ we need to find some i such that $x \ntrianglelefteq_i y$. If $x = a_j$ and $y = a_k$ for $j \neq k$ it holds that $a_j \ntrianglelefteq_j a_k$. If $x = b_j$ and $y = b_k$ for $j \neq k$ it holds that $b_j \ntrianglelefteq_k b_k$. If $x = b_j$ and $y = a_j$ it holds that $b_j \ntrianglelefteq_k a_j$ for any $k \neq j$. Finally, if $x = a_j$ and $y = b_j$ it holds that $a_j \ntrianglelefteq_j b_j$.

We conclude that $\{\trianglelefteq_0, \ldots, \trianglelefteq_{n-1}\}$ realizes F_n and so $\dim(F_n) = n$. □

If $\{\trianglelefteq_0, \ldots, \trianglelefteq_{m-1}\}$ realizes F_n for some $m > n$ then the pair (b_i, a_i) may belong to more than one linearization. On the other hand, from the proof of the Theorem 2.1.6, it seems natural that any realization of F_n consisting of exactly n linearizations is such that the pair (b_i, a_i) occurs in exactly one linearization. Indeed this is the case.

Lemma 2.1.7 (RCA_0). *For each set of linearizations* $\{\trianglelefteq_0, \ldots, \trianglelefteq_{n-1}\}$ *that realizes* F_n *and for each* $k < n$ *there exists exactly one* $i < n$ *such that* $b_k \trianglelefteq_i a_k$.

Proof. If $n = 2$ the result is trivial so we may assume $n > 2$. For each $k < n$, the existence of i is trivial since $a_k \mid b_k$.

To prove uniqueness suppose, without loss of generality, that $b_{n-1} \trianglelefteq_{n-1} a_{n-1}$ and $b_{n-1} \trianglelefteq_{n-2} a_{n-1}$. Then, by the first argument in the proof of Theorem 2.1.6, the set of restrictions of $\{\trianglelefteq_0, \dots, \trianglelefteq_{n-3}\}$ to F_{n-1} realizes F_{n-1}, against Theorem 2.1.6 itself. □

2.2 BOUNDING THEOREMS

Several theorems have been proved about the dimension of posets: for example in [Fis85] there are several statements that bound the dimension of a poset in terms of other quantities, such as its width, its height or its cardinality. We are interested in statements that give an upper bound to the dimension of a poset in terms of the dimension of its subposets, obtained by removing one or more points:

- $\mathsf{DB_p}$: for each poset $(P, \preceq)$ and each $x_0 \in P$,

$$\dim(P, \preceq) \leqslant \dim(P \setminus \{x_0\}, \preceq) + 1.$$

- $\mathsf{DBi_n}$: for each poset $(P, \preceq)$ and each set of pairwise incomparable chains $C_i \subseteq P$ for $i < n$

$$\dim(P, \preceq) \leqslant \dim(P \setminus \textstyle\bigcup_{i<n} C_i, \preceq) + \max\{2, n\}.$$

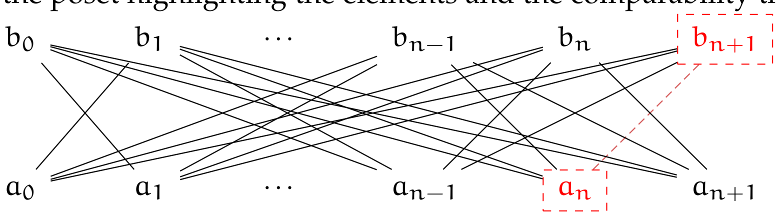
$\mathsf{DB_p}$, $\mathsf{DBi_1}$ and $\mathsf{DBi_2}$ are proved in [Hir55], while for $n \geqslant 3$ $\mathsf{DBi_n}$ is a new (to the best of our knowledge) natural extension of the previous results.

Since these theorems give upper bounds for the dimension of a poset, we focus on the case of finite dimension.

We show that each $\mathsf{DBi_n}$ is provable in, and in fact equivalent to, $\mathsf{WKL_0}$ in Section 2.3. In Section 2.4 we deal with $\mathsf{DB_p}$, showing how to prove it either in $\mathsf{WKL_0}$ or using $\mathsf{I}\Sigma^0_2$.

Starting from the posets F_n, it is fairly easy to construct posets that show that the bounds provided by $\mathsf{DBi_n}$ are sharp. In the examples we highlight the crucial comparabilities.

Example 2.2.1. Fix $n \geqslant 2$ and consider F_{n+2} and the chain $C = \{a_n, b_{n+1}\}$. The poset $F_{n+2} \setminus C$ is a copy of F_{n+1} plus the relation $a_n \prec b_n$ (it suffices to rename a_{n+1} as a_n). In the figure we show the poset highlighting the elements and the comparability that we remove.

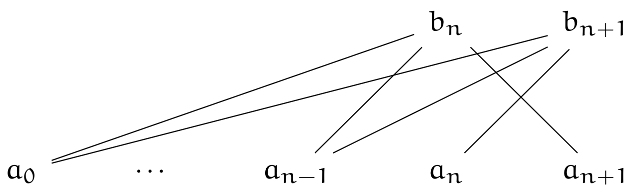


If $\trianglelefteq_0, \dots, \trianglelefteq_n$ are the $n+1$ linearizations constructed in the proof of Theorem 2.1.6 to realize F_{n+1}, it is immediate to see that $\trianglelefteq_0, \dots, \trianglelefteq_{n-1}$ suffice to realize $F_{n+2} \setminus C$. Therefore $\dim(F_{n+2} \setminus C) \leqslant n$ while $\dim(F_{n+2}) = n+2$ by Theorem 2.1.6. Therefore the bound in DBi_1 is sharp.

Example 2.2.2. Consider again F_{n+2} for $n \geqslant 2$, and let $C_0 = \{a_{n+1}, b_n\}$ and $C_1 = \{a_n, b_{n+1}\}$. C_0 and C_1 are incomparable chains, $F_{n+2} \setminus (C_0 \cup C_1)$ is exactly F_n, so that we have $\dim(F_{n+2} \setminus (C_0 \cup C_1)) = n$ and the bound in DBi_2 is sharp.

We now generalize Example 2.2.2 to DBi_n for $n > 2$. We need to adjust the construction, as in F_n there are no more than two incomparable chains of the form $\{a_i, b_j\}$.

Example 2.2.3. Fix $n \geqslant 3$. Consider the poset F_{n+2} and the n incomparable chains $C_i = \{b_i\}$ for $i < n$. Let $C = \bigcup_{i<n} C_i$: we claim that $\dim(F_{n+2} \setminus C) = 2$. In the figure we show the poset after we removed the elements.



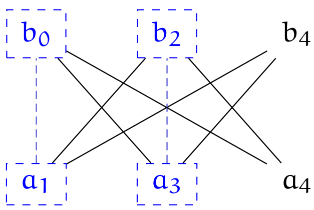
To prove that $\dim(F_{n+2} \setminus C) \leqslant 2$ we explicitly define two linearizations $\trianglelefteq_0$ and $\trianglelefteq_1$:

$$a_{n+1} \trianglelefteq_0 a_0 \trianglelefteq_0 \dots \trianglelefteq_0 a_{n-1} \trianglelefteq_0 b_n \trianglelefteq_0 a_n \trianglelefteq_0 b_{n+1},$$
$$a_n \trianglelefteq_1 a_{n-1} \trianglelefteq_1 \dots \trianglelefteq_1 a_0 \trianglelefteq_1 b_{n+1} \trianglelefteq_1 a_{n+1} \trianglelefteq_1 b_n.$$

It is routine to check that $\trianglelefteq_0$ and $\trianglelefteq_1$ realize $F_{n+2} \setminus C$.

One may wonder if the requirement in DBi_n ($n \geqslant 2$) that the chains are incomparable is necessary. The following example shows that this is the case. For simplicity, we deal with the case $n = 2$ though the construction can be adapted to any n

Example 2.2.4. Consider F_5 and the chains $C_0 = \{a_0, b_1\}$ and $C_1 = \{a_2, b_3\}$ which are comparable since $a_0 \preceq b_3$ and $a_2 \preceq b_1$. Then $F_5 \setminus (C_0 \cup C_1)$ is F_3 with two new comparabilities between a's and b's highlighted in the figure.



The linearizations

$$a_1 \trianglelefteq_0 a_3 \trianglelefteq_0 b_4 \trianglelefteq_0 a_4 \trianglelefteq_0 b_0 \trianglelefteq_0 b_2,$$
$$a_4 \trianglelefteq_1 a_3 \trianglelefteq_1 a_1 \trianglelefteq_1 b_2 \trianglelefteq_1 b_0 \trianglelefteq_1 b_4$$

realize $F_5 \setminus (C_0 \cup C_1)$. We conclude that the dimension is exactly 2 and consequently $\dim(F_5) = 5 \nleq 4 = \dim(F_5 \setminus (C_0 \cup C_1)) + 2$.

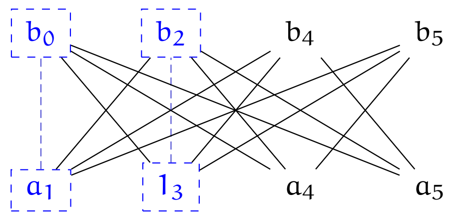
Despite Example 2.2.4, we may still state a bound without the requirement of incomparability of the chains.

- DBc_n: for each poset $(P, \preceq)$ and each family of chains $C_i \subseteq P$ for $i < n$,

$$\dim(P, \preceq) \leqslant \dim(P \setminus \bigcup_{i<n} C_i, \preceq) + 2n.$$

We show that this bound is sharp. As with Example 2.2.4, for simplicity we deal with the case $n = 2$.

Example 2.2.5. Consider F_6, $C_0 = \{a_0, b_1\}$ and $C_1 = \{a_2, b_3\}$. Then $F_6 \setminus (C_0 \cup C_1)$ is F_4 with two comparabilities between the a's and the b's highlighted in the figure.



The linearizations

$$a_5 \trianglelefteq_0 a_3 \trianglelefteq_0 a_1 \trianglelefteq_0 b_4 \trianglelefteq_0 a_4 \trianglelefteq_0 b_0 \trianglelefteq_0 b_2 \trianglelefteq_0 b_5,$$
$$a_4 \trianglelefteq_1 a_1 \trianglelefteq_1 a_3 \trianglelefteq_1 b_5 \trianglelefteq_1 a_5 \trianglelefteq_1 b_2 \trianglelefteq_1 b_0 \trianglelefteq_1 b_4$$

realize $F_6 \setminus (C_0 \cup C_1)$ so that $\dim(F_6) = 6 = \dim(F_6 \setminus (C_1 \cup C_2)) + 4$.

A simple proof of $\forall n\, \mathrm{DBc}_n$ can be obtained by applying repeatedly DBi_1. However this proof cannot be formalized in WKL_0, and in Section 2.3 we give a direct proof.

2.3 DBi_n AND DBc_n

In this section we deal with DBi_n and DBc_n. We show that for each n, each of these dimension bounds is equivalent to WKL_0. The following notion is the basic tool for proving these statements.

Definition 2.3.1. Let $(P, \preceq)$ be a poset and let $C^0, C^1 \subseteq P$ be incomparable chains. We say that a linearization $\trianglelefteq$ *puts* C^0 *at the bottom and* C^1 *at the top of* P if for each $x \in P$, $c_0 \in C^0$ and $c_1 \in C^1$, if $x \mid c_0$ and $x \mid c_1$ then $c_0 \trianglelefteq x \trianglelefteq c_1$.

Lemma 2.3.2 (WKL$_0$). *Let* $(P_i, \preceq_i)_{i \in \mathbb{N}}$ *be a sequence of partial orders and let* $(C_i^j)_{i \in \mathbb{N}, j<2}$ *be such that for each* i C_i^0 *and* C_i^1 *are incomparable chains in* P_i*. There exists a sequence* $(P_i, \trianglelefteq_i)_{i \in \mathbb{N}}$ *of linearizations of* $(P_i, \preceq_i)_{i \in \mathbb{N}}$ *such that each* $\trianglelefteq_i$ *puts* C_i^0 *at the bottom and* C_i^1 *at the top of* P_i.

Proof. For each $i \in \mathbb{N}$ define R_i^0 and R_i^1 over P_i as follows:

$$x R_i^0 y \text{ if and only if } x \mid y \land x \in C_i^0 \land y \notin C_i^0,$$
$$x R_i^1 y \text{ if and only if } x \mid y \land x \notin C_i^1 \land y \in C_i^1.$$

We claim that each $R_i = \preceq_i \cup R_i^0 \cup R_i^1$ is acyclic. Assume that

$$x_0 R_i x_1 R_i \dots R_i x_{m-1} R_i x_0.$$

Since $\preceq_i$ is transitive, we may assume that it does not occur twice consecutively. It is immediate that also each of R_i^0 or R_i^1 cannot occur twice consecutively. Since the chains are incomparable we cannot have $x_i R_i^1 x_{i+1} R_i^0 x_{i+2}$ and $x_i R_i^1 x_{i+1} \preceq_i x_{i+2} R_i^0 x_{i+3}$ as well.

If R_i^1 occurs in the cycle, up to renaming its elements, we may assume that $x_0 R_i^1 x_1 \preceq_i x_2 \dots$ and then R_i^0 does not occur. Therefore $\preceq_i$ and R_i^1 alternate in the cycle and this entails that m is even. Hence, the cycle has the form

$$x_0 R_i^1 x_1 \preceq_i \dots R_i^1 x_{m-1} \preceq_i x_0$$

For each $j < \frac{m}{2}$ we have that $x_{2j} \notin C_i^1$ while $x_{2j+1} \in C_i^1$. We claim that for each $j < \frac{m}{2} - 1$, $x_{2j+1} \preceq_i x_{2j+3}$. Indeed $x_{2j+1} \preceq_i x_{2j+2}$ and $x_{2j+2} \mid x_{2j+3}$. Since $x_{2j+1}, x_{2j+3} \in C_i^1$ it must be $x_{2j+1} \preceq_i x_{2j+3}$. Thus $x_1 \preceq_i x_{m-1} \preceq_i x_0$, contradicting $x_0 \mid x_1$.

If R_i^1 does not occur in the cycle, then R_i^0 does and an analogous argument leads to a contradiction, completing the proof of the claim.

By Corollary 2.1.4, which is the step where we use WKL$_0$, and by Theorem 2.1.1 the relations $(R_i)_{i \in \mathbb{N}}$ can be extended to linear orders $(\trianglelefteq_i)_{i \in \mathbb{N}}$ which have the prescribed properties. □

Notice that we include the case in which one of the chains is empty.

Lemma 2.3.2 can be reversed. The reversal obtained is sharper since we only use a single partial order.

Theorem 2.3.3 (RCA$_0$). *The following are equivalent:*

(1) WKL_0,

(2) *if* $(P, \preceq)$ *is a poset and* $C_0, C_1 \subseteq P$ *are incomparable chains, there exists a linearization* $(P, \trianglelefteq)$ *of* $(P, \preceq)$ *that puts* C_0 *at the bottom and* C_1 *at the top of* P.

Proof. Lemma 2.3.2 proves that (1) implies (2).

For the converse let f, g be 1-1 functions with disjoint ranges and let $P = \{z\} \cup \{x_n, a_n, b_n : n \in \mathbb{N}\}$. We define a partial order $\preceq$ on P as follows:

- $a_i \preceq a_j$ if and only if $i < j$;
- $b_i \preceq b_j$ if and only if $j < i$;
- $x_n \preceq a_i$ if and only if $\exists j < i + 1\, f(j) = n$;
- $b_i \preceq x_n$ if and only if $\exists j < i + 1\, g(j) = n$.

Let $C_0 = \{a_n : n \in \mathbb{N}\}$ and $C_1 = \{b_n : n \in \mathbb{N}\}$ which are incomparable chains. By (2) let $\trianglelefteq$ be a linearization of $\preceq$ that puts C_0 at the bottom and C_1 at the top of P. The set $\{n : x_n \trianglelefteq z\}$ separates $\operatorname{ran}(f)$ and $\operatorname{ran}(g)$. □

We now deal with the bound DBi_n and show that it is provable in WKL_0. As we mentioned in Section 2.2, we only deal with posets with finite dimension.

Theorem 2.3.4 (WKL_0)**.** $\forall n\, \mathsf{DBi}_n$.

Proof. Cases $n = 0$ and $n = 1$ are trivial. Let $(P, \preceq)$ be a poset, fix $n > 1$ and a set of pairwise incomparable chains $C_i \subseteq P$ for $i < n$. Let $C = \bigcup_{i<n} C_i$. Our goal is to prove that $\dim(P, \preceq) \leqslant \dim(P \setminus C, \preceq) + n$.

Let $m = \dim(P \setminus C, \preceq)$ and fix a set of linearizations $\{\trianglelefteq_0, \dots, \trianglelefteq_{m-1}\}$ realizing $P \setminus C$. We construct a set of $m + n$ linear orders which realizes P.

For each $i < m$ let $\preceq_i^* = \preceq \cup \trianglelefteq_i$. We claim that each $\preceq_i^*$ is acyclic. Towards a contradiction, let $x_0, \dots, x_{\ell-1}$ be such that

$$x_0 \preceq_i^* x_1 \preceq_i^* \dots \preceq_i^* x_{\ell-1} \preceq_i^* x_0.$$

Since both $\preceq$ and $\trianglelefteq_i$ are transitive relations we may assume that ℓ is even

$$x_0 \preceq x_1 \trianglelefteq_i \dots \preceq x_{\ell-1} \trianglelefteq_i x_0.$$

Since $\trianglelefteq_i$ is a relation over $P \setminus C$, it follows that for each $j < \ell$ $x_j \in P \setminus C$. By the fact that $\trianglelefteq_i$ extends $\preceq$ on its domain, we may replace each occurrence of $\preceq$ with $\trianglelefteq_i$ obtaining a cycle with respect to $\trianglelefteq_i$, a contradiction.

By Corollary 2.1.4 and Theorem 2.1.1 each $\preceq_i^*$ can be extended to a linear order $\trianglelefteq_i^*$.

Next we build n further linearizations to deal with the chains C_i for $i < n$. We apply Lemma 2.3.2 to the n pairs of chains $C_{[j]_n}, C_{[j+1]_n}$ for $j < n$, where $[j]_n$ the residue class of j modulo n. We obtain a sequence of linear orders $(\trianglelefteq_{m+j}^*)_{j<n}$, each extending $(P, \preceq)$ and such that $\trianglelefteq_{m+j}^*$ puts $C_{[j]_n}$ at the bottom and $C_{[j+1]_n}$ at the top of P.

We are left to prove that the set $\{\trianglelefteq_0^*, \dots, \trianglelefteq_{m+n-1}^*\}$ realizes $(P, \preceq)$. As we noticed at the end of Section 1.2 it suffices to prove that for each $x, y \in P$ such that $x \mid y$, there exists $i < m + n$ such that $x \ntrianglelefteq_i^* y$. We distinguish some cases:

(1) if $x \in C_i$ then $\trianglelefteq_{m+[i-1]_n}^*$, which puts C_i at the top of P, works;

(2) if $y \in C_i$ then $\trianglelefteq_{m+[i]_n}^*$, which puts C_i at the bottom of P, works;

(3) if $x, y \in P \setminus C$ then recall that $\{\trianglelefteq_0, \dots, \trianglelefteq_{m-1}\}$ realizes $P \setminus C$; hence there exists $i < n$ such that $x \ntrianglelefteq_i y$ and consequently $\trianglelefteq_i^*$, which extends $\trianglelefteq_i$, works.

Therefore $\dim(P, \preceq) \leqslant \dim(P \setminus C, \preceq) + n$. □

Provability of DBi_1 in WKL_0 now follows immediately.

Corollary 2.3.5 (WKL_0)**.** DBi_1.

Proof. Let $(P, \preceq)$ be a poset and $C \subseteq P$ a chain. Then, applying Theorem 2.3.4 with $n = 2$, $C_0 = C$ and $C_1 = \emptyset$, we have $\dim(P, \preceq) \leqslant \dim(P \setminus C, \preceq) + 2$. □

The last bound we are left to prove is DBc_n. As WKL_0 has limited induction, we cannot carry out the straightforward inductive proof applying DBi_1 (which coincides with DBc_1) repeatedly.

Theorem 2.3.6 (WKL_0)**.** $\forall n\, \mathsf{DBc}_n$.

Proof. Let $(P, \preceq)$ be a poset. For $i < n$ let $C_i \subseteq P$ be chains and $C = \bigcup_{i<n} C_i$. We need to prove that $\dim(P, \preceq) \leqslant \dim(P \setminus C, \preceq) + 2n$.

Let $m = \dim(P \setminus C, \preceq)$ and fix a set of linearizations $\{\trianglelefteq_0, \dots, \trianglelefteq_{m-1}\}$ which realizes $(P \setminus C, \preceq)$. We construct a set of $m + 2n$ linear orders which realizes $(P, \preceq)$.

For each $i < m$ we extend $\trianglelefteq_i$ to a linearization $\trianglelefteq_i^*$ of $(P, \preceq)$ as we did in Theorem 2.3.4.

We then apply Lemma 2.3.2 to the poset P and to the $2n$ pairs of chains $C_i, \emptyset$ and $\emptyset, C_i$ for $i < n$. We obtain linearizations $\{\trianglelefteq_{m+j}^*\}_{j<2n}$ such that if $j < 2n$ is even (respectively, odd) then $\trianglelefteq_{m+j}^*$ is the linearization of P that puts $C_{\frac{i}{2}}$ at the bottom (respectively, that puts $C_{\frac{i-1}{2}}$ at the top).

Showing that the set $\{\trianglelefteq_0^*, \dots, \trianglelefteq_{m+2n-1}^*\}$ realizes $(P, \preceq)$ is similar to the analogous proof in Theorem 2.3.4. □

Now we deal with the reversals.

Theorem 2.3.7 (RCA_0)**.** *The following are equivalent:*

(1) WKL_0,

(2) $\forall n\, \mathsf{DBi}_n$,

(3) DBi_2,

(4) $\forall n\, \mathsf{DBc}_n$,

(5) DBi_1.

Proof. Theorem 2.3.4 proves that (1) implies (2), while Theorem 2.3.6 shows that (1) implies (4). (2) implies (3) and (4) implies (5) are obvious (notice that DBi_1 and DBc_1 coincide). The proof of Corollary 2.3.5 shows that (3) implies (5). Therefore we are left to prove that (5) implies (1).

Let f, g be 1-1 functions with disjoint ranges: we want to show that there exists a set A such that $\mathrm{ran}(f) \subseteq A$ and $A \cap \mathrm{ran}(g) = \emptyset$. Let

$$P = \{x^i, y^i : i \in \mathbb{N}\} \cup \{c^r_j, d^r_j : j < 3, r \in \mathbb{N}\} \cup \{p^s_j, q^s_j : j < 3, s \in \mathbb{N}\}.$$

To define a partial order $\preceq$ on P we start with a level function $\ell \colon P \to \mathbb{N}$ defined by

$$\ell(z) = \begin{cases} i & \text{if } z = x^i, y^i; \\ f(r) & \text{if } z = c^r_j, d^r_j; \\ g(s) & \text{if } z = p^s_j, q^s_j. \end{cases}$$

Notice that RCA_0 suffices to prove that ℓ exists. Let $P_n = \{z \in P : \ell(z) = n\}$: we say that P_n is a level of P. Since f is 1-1, c^r_j and $d^{r'}_k$ belong to the same level if and only if $r = r'$, and the same holds for p^s_j and $q^{s'}_k$. Furthermore we have $\ell(c^r_j) \neq \ell(p^s_k)$ for all r, s, j and k.

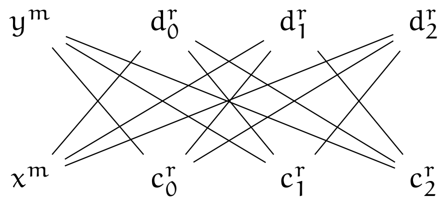
We define the strict partial order $\prec$ by setting for each $u, v \in P$:

- if $\ell(u) < \ell(v)$ then $u \prec v$,
- if $\ell(u) = \ell(v) = m$ then $u \prec v$ if and only if one of the following alternatives holds

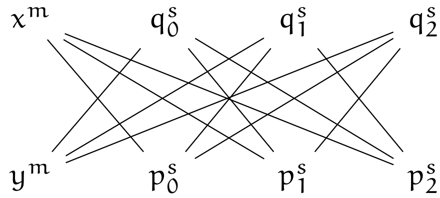
$$\begin{aligned} u = x^m, v = d^r_j, f(r) = m &\qquad u = p^s_j, v = x^m, g(s) = m \\ u = c^r_j, v = y^m, f(r) = m &\qquad u = y^m, v = q^s_j, g(s) = m \\ u = c^r_j, v = d^r_{j'}, j \neq j', f(r) = m &\qquad u = p^s_j, v = q^s_{j'}, j \neq j', g(s) = m. \end{aligned}$$

To check that $(P, \preceq)$ is a partial order, notice that asymmetry is trivial, while transitivity is immediate when the elements belong to different levels (because the levels are linearly ordered by $\preceq$ as the natural numbers) and follows easily in the other case because no level has chains of length 3.

If $m \notin \mathrm{ran}(f) \cup \mathrm{ran}(g)$ then $P_m = \{x^m, y^m\}$ is an antichain. If $f(r) = m$ the level P_m is a copy of F_4 where $\{x^m, c_0^r, c_1^r, c_2^r\}$ is the set of the a_i's while $\{y^m, d_0^r, d_1^r, d_2^r\}$ is the set of the b_i's (recall the notation of Definition 2.1.5). The figure below illustrates level m.

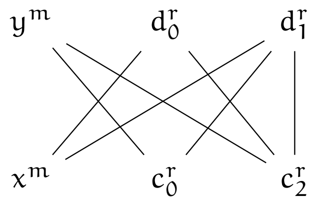


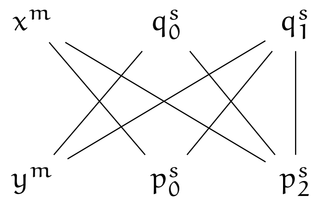
Similarly, if $g(s) = m$ the level P_m is a copy of F_4 where $\{y^m, p_0^s, p_1^s, p_2^s\}$ is the set of the a_i's while $\{x^m, q_0^s, q_1^s, q_2^s\}$ is the set of the b_i's. The figure below illustrates level m in this case.



Even though the dimension of each level is at most 4, in RCA_0 we cannot define a set of four linearizations of $\preceq$ that witnesses $\dim(P, \preceq) = 4$. Indeed, to do so we need to know whether $m \in \mathrm{ran}(f)$ (in which case Lemma 2.1.7 implies that three of the linearizations need to put x^m below y^m) or $m \in \mathrm{ran}(g)$ (in which case three of the linearizations need to put x^m above y^m).

Let $C = \{c_1^r, d_2^r, p_1^s, q_2^s : r, s \in \mathbb{N}\}$ and notice that it is a chain. We now consider the poset $(P \setminus C, \preceq)$. If $m \in \mathrm{ran}(f)$ or $m \in \mathrm{ran}(g)$ then $P_m \setminus C$ has one of the following forms.





We claim that $\dim(P \setminus C, \preceq) = 2$. First notice that for each m we have that $x^m, y^m \notin C$ and $x^m \mid y^m$. Therefore $\dim(P \setminus C, \preceq) \geqslant 2$. We now exhibit two linearizations $\trianglelefteq_0$ and $\trianglelefteq_1$ that realize $(P \setminus C, \preceq)$. We define $\trianglelefteq_0$ by:

- if $u \preceq v$ then $u \trianglelefteq_0 v$,
- $x^m \trianglelefteq_0 y^m$,
- for each r if $f(r) = m$ then $x^m \trianglelefteq_0 c_2^r \trianglelefteq_0 d_0^r \trianglelefteq_0 c_0^r \trianglelefteq_0 d_1^r \trianglelefteq_0 y^m$,
- for each s if $g(s) = m$ then $p_0^s \trianglelefteq_0 p_2^s \trianglelefteq_0 x^m \trianglelefteq_0 y^m \trianglelefteq_0 q_1^s \trianglelefteq_0 q_0^s$.

In a similar fashion, we define $\trianglelefteq_1$:

- if $u \preceq v$ then $u \trianglelefteq_1 v$,
- $y^m \trianglelefteq_1 x^m$,
- for each r if $f(r) = m$ then $c_0^r \trianglelefteq_1 c_2^r \trianglelefteq_1 y^m \trianglelefteq_1 x^m \trianglelefteq_1 d_1^r \trianglelefteq_1 d_0^r$,
- for each s if $g(s) = m$ then $y^m \trianglelefteq_1 p_2^s \trianglelefteq_1 q_0^s \trianglelefteq_1 p_0^s \trianglelefteq_1 q_1^s \trianglelefteq_1 x^m$.

To prove that $\trianglelefteq_0$ and $\trianglelefteq_1$ realize the poset, we need to show that if $u \mid v$ then either $u \ntrianglelefteq_0 v$ or $u \ntrianglelefteq_1 v$. Since $u \mid v$ implies $\ell(u) = \ell(v)$ there are at most 16 cases, that are easily checked.

$$
\begin{array}{ll}
(y^n, x^n) \notin \trianglelefteq_0 & (x^n, y^n) \notin \trianglelefteq_1 \\
(c_0^r, d_0^r) \notin \trianglelefteq_0, (d_0^r, c_0^r) \notin \trianglelefteq_1 & (q_0^s, p_0^s) \notin \trianglelefteq_0, (p_0^s, q_0^s) \notin \trianglelefteq_1 \\
(c_0^r, x^n) \notin \trianglelefteq_0, (x^n, c_0^r) \notin \trianglelefteq_1 & (y^n, p_0^s) \notin \trianglelefteq_0, (p_0^s, y^n) \notin \trianglelefteq_1 \\
(c_2^r, x^n) \notin \trianglelefteq_0, (x^n, c_2^r) \notin \trianglelefteq_1 & (y^n, p_2^s) \notin \trianglelefteq_0, (p_2^s, y^n) \notin \trianglelefteq_1
\end{array}
$$

$$
\begin{array}{ll}
(c_0^r, c_2^r) \notin \trianglelefteq_0, (c_2^r, c_0^r) \notin \trianglelefteq_1 & (p_2^s, p_0^s) \notin \trianglelefteq_0, (p_0^s, p_2^s) \notin \trianglelefteq_1 \\
(y^n, d_0^r) \notin \trianglelefteq_0, (d_0^r, y^n) \notin \trianglelefteq_1 & (q_0^s, x^n) \notin \trianglelefteq_0, (x^n, q_0^s) \notin \trianglelefteq_1 \\
(y^n, d_1^r) \notin \trianglelefteq_0, (d_1^r, y^n) \notin \trianglelefteq_1 & (q_1^s, x^n) \notin \trianglelefteq_0, (x^n, q_1^s) \notin \trianglelefteq_1 \\
(d_1^r, d_0^r) \notin \trianglelefteq_0, (d_0^r, d_1^r) \notin \trianglelefteq_1 & (q_0^s, q_1^s) \notin \trianglelefteq_0, (q_1^s, q_0^s) \notin \trianglelefteq_1
\end{array}
$$

(5) implies that $\dim(P, \preceq) \leqslant \dim(P \setminus C, \preceq) + 2 = 4$. For this reason we can fix a set of linearizations $\{\trianglelefteq_0^*, \trianglelefteq_1^*, \trianglelefteq_2^*, \trianglelefteq_3^*\}$ which realizes $(P, \preceq)$. Notice that for each $i < 4$, the restriction of $\trianglelefteq_i^*$ to the level P_m is a linearization of $(P_m, \preceq)$ and a fortiori the set $\{\trianglelefteq_0^*, \trianglelefteq_1^*, \trianglelefteq_2^*, \trianglelefteq_3^*\}$ restricted to P_m realizes $(P_m, \preceq)$. We already noticed that P_m is either an antichain of two elements or a copy of F_4. By Lemma 2.1.7 if $m \in \operatorname{ran}(f)$ then $|\{i < 4 : y^m \trianglelefteq_i^* x^m\}| = 1$ while if $m \in \operatorname{ran}(g)$ then $|\{i < 4 : y^m \trianglelefteq_i^* x^m\}| = 3$. If $m \notin \operatorname{ran}(f) \cup \operatorname{ran}(g)$ we only know that $1 \leqslant |\{i < 4 : y^m \trianglelefteq_i^* x^m\}| \leqslant 3$. If we let

$$A = \{m : |\{i < 4 : y^m \trianglelefteq_i^* x^m\}| = 1\}$$

we have $\mathrm{ran}(f) \subseteq A$ and $A \cap \mathrm{ran}(g) = \emptyset$, as desired. $\square$

The key point of the construction in Theorem 2.3.7 is that $\dim(P \setminus C, \preceq) = 2$. Therefore, when we add back the chain C and apply DBi_1, we obtain a set of four linearizations which realizes P and we can exploit Lemma 2.1.7 to define the separator set A. The same construction can be used to prove that DBi_3 is equivalent to WKL_0 too.

Corollary 2.3.8 (RCA_0). WKL_0 *is equivalent to* DBi_3.

Proof. The forward direction is proved in Theorem 2.3.4.

For the backward direction, given f and g we consider the same poset $(P, \preceq)$ of the proof of Theorem 2.3.7. Let C_0 be the chain called C in that proof and let $C_1 = C_2 = \emptyset$ (regarded as chains). The poset $(P \setminus (C_0 \cup C_1 \cup C_2), \preceq)$ coincides with $(P \setminus C, \preceq)$ and has dimension 2. By DBi_3 we get that $\dim(P, \preceq) \leqslant 5$ and we can fix a set $\{\trianglelefteq_0, \trianglelefteq_1, \trianglelefteq_2, \trianglelefteq_3, \trianglelefteq_4\}$ of linearizations which realizes $(P, \preceq)$.

If $f(r) = m$ for some r, then three distinct linearizations have to deal with the incomparabilities $c_0^r \mid d_0^r$, $c_1^r \mid d_1^r$ and $c_2^r \mid d_2^r$ in P_m. This means that if $m \in \mathrm{ran}(f)$ then $|\{i < 5 : y^m \trianglelefteq_i^* x^m\}| \leqslant 2$. Analogously if $m \in \mathrm{ran}(g)$ then $|\{i < 5 : y^m \trianglelefteq_i^* x^m\}| \geqslant 3$. Therefore we can define the separator set A as the set of $m \in \mathbb{N}$ such that $|\{i < 5 : y^m \trianglelefteq_i^* x^m\}| \leqslant 2$. $\square$

The idea of Theorem 2.3.7 can be further exploited to obtain a reversal for every DBc_n.

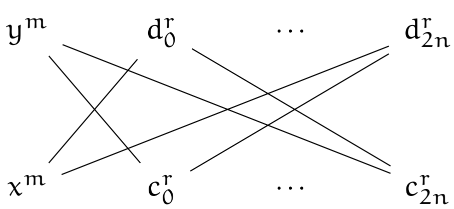
Theorem 2.3.9. *For each* n, RCA_0 *proves the equivalence between* DBc_n *and* WKL_0.

Proof. Theorem 2.3.6 shows one implication, so we deal with the converse.

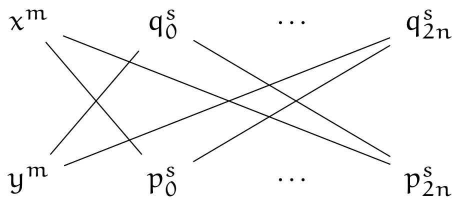
If $n = 1$ then DBc_1 is DBi_1 and the implication was proved in Theorem 2.3.7. So fix $n > 1$ and as usual let f, g be 1-1 functions with disjoint ranges. Let

$$P = \{x^i, y^i : i \in \mathbb{N}\} \cup \{c_j^r, d_j^r : j < 2n+1, r \in \mathbb{N}\} \cup \{p_j^s, q_j^s : j < 2n+1, s \in \mathbb{N}\}.$$

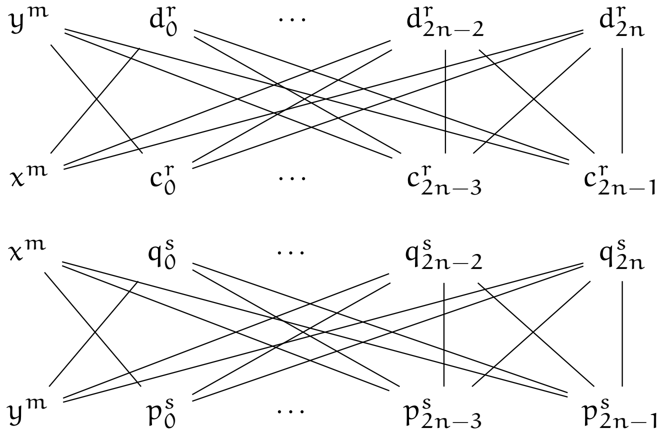
We define the partial order $\preceq$ by a construction similar to the one used in the proof of Theorem 2.3.7. We define a level function $\ell : P \to \mathbb{N}$ as before and we let $P_m = \{z \in P : \ell(z) = m\}$. If $m \notin \mathrm{ran}(f) \cup \mathrm{ran}(g)$ then P_m is an antichain consisting of the elements x^m and y^m. If $m \in \mathrm{ran}(f)$ there exists a unique r such that $f(r) = m$ and then P_m is a copy of F_{2n+2} where $\{x^m, c_j^r : j < 2n+1\}$ is the set of the a_i's while $\{y^m, d_j^r : j < 2n+1\}$ is the set of the b_i's. The figure below illustrates a level of this form.



Similarly, if $m \in \operatorname{ran}(g)$ there exists a unique s such that $g(s) = m$ and then P_m is a copy of F_{2n+2} where $\{y^m, p_j^s : j < 2n+1\}$ is the set of the a_i's while $\{x^m, q_j^s : j < 2n+1\}$ is the set of the b_i's. The figure below illustrates a level of this form.



For $1 \leqslant i \leqslant n$ consider the chain $C_i = \{c_{2i}^r, d_{2i-1}^r, p_{2i}^s, q_{2i-1}^s : r, s \in \mathbb{N}\}$. Let $C = \bigcup_{i=1}^n C_i$. If $m \in \operatorname{ran}(f)$ or $m \in \operatorname{ran}(g)$ then $P_m \setminus C$ has one of the following forms.



We claim that $\dim(P \setminus C, \preceq) = 2$. First notice that $P \setminus C$ is not a chain which means that $\dim(P \setminus C, \preceq) \geqslant 2$. Hence it suffices to exhibit two linearizations $\trianglelefteq_0$ and $\trianglelefteq_1$ that realize $(P \setminus C, \preceq)$. We define $\trianglelefteq_0$ by:

- if $z_1 \preceq z_2$ then $z_1 \trianglelefteq_0 z_2$,
- $x^m \trianglelefteq_0 y^m$,
- for each r if $f(r) = m$ then

$$x^m \trianglelefteq_0 c_{2n-1}^r \trianglelefteq_0 \ldots \trianglelefteq_0 c_1^r \trianglelefteq_0 d_0^r \trianglelefteq_0 c_0^r \trianglelefteq_0 d_2^r \trianglelefteq_0 \ldots \trianglelefteq_0 d_{2n}^r \trianglelefteq_0 y^m,$$

- for each s if $g(s) = m$ then

$$p_0^s \trianglelefteq_0 \ldots \trianglelefteq_0 p_{2n-1}^s \trianglelefteq_0 x^m \trianglelefteq_0 y^m \trianglelefteq_0 q_{2n}^s \trianglelefteq_0 \ldots \trianglelefteq_0 q_0^s.$$

In a similar fashion, we define $\trianglelefteq_1$:

- if $z_1 \preceq z_2$ then $z_1 \trianglelefteq_1 z_2$,
- $y^m \trianglelefteq_1 x^m$,
- for each r if $f(r) = m$ then

$$c_0^r \trianglelefteq_1 \ldots \trianglelefteq_1 c_{2n-1}^r \trianglelefteq_1 y^m \trianglelefteq_1 x^m \trianglelefteq_1 d_{2n}^r \trianglelefteq_1 \ldots \trianglelefteq_1 d_0^r,$$

- for each s if $g(s) = m$ then

$$y^m \trianglelefteq_0 p_{2n-1}^s \trianglelefteq_0 \ldots \trianglelefteq_0 p_1^s \trianglelefteq_0 q_0^s \trianglelefteq_0 p_0^s \trianglelefteq_0 q_2^s \trianglelefteq_0 \ldots \trianglelefteq_0 q_{2n}^s \trianglelefteq_0 y^m.$$

It is easy to verify that $\trianglelefteq_0$ and $\trianglelefteq_1$ realize $(P \setminus C, \preceq)$.

By DBc_n we know that $\dim(P, \preceq) \leqslant 2n+2$ and we can fix a set of linearizations $\{\trianglelefteq_i^* : i < 2n+2\}$ which realizes $(P, \preceq)$. Notice that for each $i < 2n+2$ and each m, the restriction of $\trianglelefteq_i^*$ to P_m is a linearization of P_m and a fortiori the set $\{\trianglelefteq_i^* : i < 2n+2\}$ restricted to P_m realizes P_m. We already noticed that P_m is either an antichain consisting of two elements or a copy of F_{2n+2}. Applying Lemma 2.1.7 as in Theorem 2.3.7, we obtain that $A = \{m : |\{i < 2n+2 : y^m \trianglelefteq_i^* x^m\}| = 1\}$ is the required separator set. □

We still need to show that DBi_n for $n \geqslant 4$ implies WKL_0. The construction of the proof of Theorem 2.3.9 cannot be used because in that poset if a chain intersects two or more levels then it is comparable with every other nonempty chain. We need a crucial modification in the construction of the poset.

Theorem 2.3.10. *For each n, RCA_0 proves the equivalence between DBi_n and WKL_0.*

Proof. Theorem 2.3.4 shows one implication, so we deal with the converse.

We already proved the reversal for $n < 4$ in Theorem 2.3.7 and Corollary 2.3.8. The proof we are about to give works for $n \geqslant 3$. As usual let f and g be 1-1 functions with disjoint ranges. Let

$$P = \{x^i, y^i : i \in \mathbb{N}\} \cup \{c_j^r, d_j^r : j < n, r \in \mathbb{N}\} \cup \{p_j^s, q_j^s : j < n, s \in \mathbb{N}\}$$

and define $\ell : P \to \mathbb{N}$ as in Theorem 2.3.7. As before, we let P_m be the level of elements z such that $\ell(z) = m$.

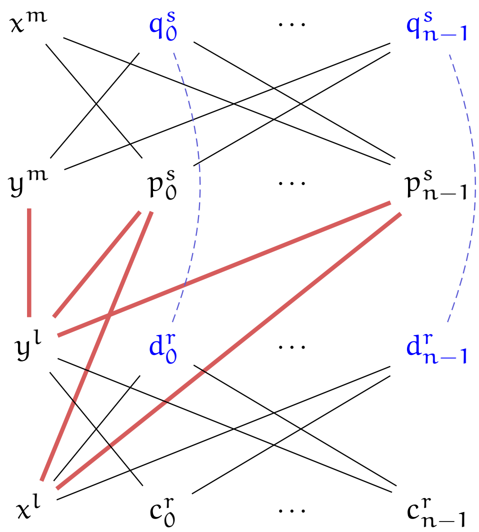
For each $z_1, z_2 \in P$ we define the partial order $\prec$ as follows.

- If neither z_1 nor z_2 is a d_j^r or a q_k^s, then $z_1 \prec z_2$ if and only if either $\ell(z_1) < \ell(z_2)$ or $z_1 = c_j^r$ and $z_2 = y^{f(r)}$ or $z_1 = p_j^r$ and $z_2 = x^{f(r)}$ (this case coincides with what we did in the previous proofs).

- If $z_1 = d^r_j$ then $z_1 \prec z_2$ if and only if $z_2 = d^k_j$ for some k such that $f(r) < f(k)$ or $z_2 = q^s_j$ for some s such that $f(r) < g(s)$ (namely, z_2 is a d^k_j or a q^s_j for the same j of z_1 and belongs to some higher level). If $z_1 = q^s_j$ the definition is analogous.

- If $z_2 = d^r_j$ then $z_1 \prec z_2$ if and only if z_1 is either $x^{f(r)}$ or c^r_i for $i \neq j$ or $\ell(z_1) < f(r)$ and if z_1 is a d^v_i or a q^s_i then $i = j$. If $z_2 = q^s_j$ the definition is analogous, replacing x^m with y^m and c^r_i with p^s_i.

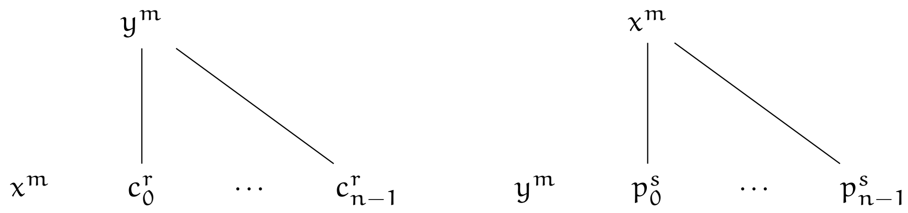
Each level P_m is either an antichain of two elements or a copy of F_{n+1}. The difference with the previous constructions lies in the comparabilities between different levels as shown in the figure below, where we are assuming that $f(r) = l$ and $g(s) = m$. We highlight with a thick red line some of the comparabilities between different levels that coincide with the posets in the proofs of Theorem 2.3.7, Corollary 2.3.8 and Theorem 2.3.9. We highlight with a dashed blue line some of the comparabilities between d^r_i's and q^s_j's: notice that $d^r_0 \not\prec p^s_0$ even though $\ell(d^r_0) < \ell(p^s_0)$.



We show that $(P, \preceq)$ is a partial order. Irreflexivity and antisymmetry are trivial. For transitivity, we pick $z_1 \prec z_2$ and $z_2 \prec z_3$ and we proceed by cases to prove that $z_1 \prec z_3$. The only interesting case is when either d^r_j or q^s_j occur among z_1, z_2, z_3 because otherwise the relation is the same as in the previous proofs. If $z_1 = d^r_j$ (the case $z_1 = q^s_j$ being analogous) then by definition also z_2 and z_3 are either d^k_j or q^s_j and so we immediately get $z_1 \prec z_3$ by the level function. If z_1 is different from each d^r_i, q^s_i and $z_2 = d^r_j$ (the case $z_2 = q^s_j$ is analogous) then z_3 must be either d^k_j for some k such that $f(r) < f(k)$ or q^s_j for some s such that $f(r) < g(s)$. On the other hand, it must be $l(z_1) \leqslant f(r)$ by definition and so again $z_1 \prec z_3$ by the level function. Finally, if z_1, z_2 are different from d^r_i, q^s_i and $z_3 = d^r_j$ (the case $z_3 = q^s_j$ being analogous) then $l(z_1) \leqslant l(z_2) \leqslant f(r)$. If $l(z_2) = f(r)$ then it must be $z_2 = x^{f(r)}$

or $z_2 = c_i^r$ for some $i \neq j$ and so $l(z_1) < l(z_2)$ while if $l(z_2) < f(r)$ then $l(z_1) < f(r)$. In both cases, the conclusion follows by the level function as before. Hence $(P, \preceq)$ is a poset.

For each $j < n$ consider the chain $C_j = \{d_j^r, q_j^s : r, s \in \mathbb{N}\}$. By definition of $\preceq$ these chains are incomparable. Let $C = \bigcup_{j<n} C_j$ and consider the poset $(P \setminus C, \preceq)$. If $m \in \operatorname{ran}(f)$ or $m \in \operatorname{ran}(g)$ then $P_m \setminus C$ has one of the following forms.



It is easy to check that $\dim(P \setminus C, \preceq) = 2$. By DBi_n we have that $\dim(P, \preceq) \leqslant n+2$ and we can fix a set $\{\trianglelefteq_0^*, \ldots, \trianglelefteq_{n+1}^*\}$ of $n+2$ linearizations which realizes $(P, \preceq)$. If $m \in \operatorname{ran}(f)$, then it must be $|\{i < n+2 : y^m \trianglelefteq_i^* x^m\}| \leqslant 2$ and analogously, if $m \in \operatorname{ran}(g)$, then it must be $|\{i < n+2 : y^m \trianglelefteq_i^* x^m\}| \geqslant n$. Since we are assuming $n \geqslant 3$, the set of $m \in \mathbb{N}$ such that $|\{i < n : y^m \trianglelefteq_i^* x^m\}| \leqslant 2$ separates $\operatorname{ran}(f)$ and $\operatorname{ran}(g)$. □

2.4 DB_p

We now deal with DB_p which was introduced in Section 2.2. We stratify the statement as follows.

- n-DB_p: for each poset $(P, \preceq)$ and each $x_0 \in P$, if we have that $\dim(P \setminus \{x_0\}, \preceq) = n$ then $\dim(P, \preceq) \leqslant n+1$.

Notice that the notation above does make sense only if $n > 0$. Clearly DB_p is the statement $\forall n(n\text{-}\mathsf{DB}_p)$.

Let $(P, \preceq)$ be a poset and $I, F \subseteq P$. We write $I \prec F$ if for each $i \in I$ and each $f \in F$ it holds that $i \prec f$. An initial interval of $(P, \preceq)$ is a $\preceq$-downward closed set. An initial interval $B \subseteq P$ that contains I and is disjoint from F is called a separator set for P, I, F. An element $b \in P$ such that $\forall i \in I \forall f \in \mathcal{F}(i \preceq b \preceq f)$ is called a separator element for P, I, F. For simplicity we just say that B is a separator set and b is a separator element without specifying the order and the sets considered when these are clear from the context. Notice that the definition of a separator element is Π_1^0.

Lemma 2.4.1 (RCA_0)**.** *Let $(P, \trianglelefteq)$ be a linear order and let $I, F \subseteq P$ such that $I \triangleleft F$. Then there exists a separator set for P, I, F.*

Proof. Suppose that there exists a separator element $b \in P$. If $b \in F$ then let $B = \{x \in P : x \lhd b\}$, otherwise let $B = \{x \in P : x \unlhd b\}$. Then B exists in RCA_0 and has the desired properties.

Otherwise, let $B = \{x \in P : \exists i \in I(x \unlhd i)\}$. Since there is no separator element, then $x \in B$ if and only if $\forall f \in F(x \lhd f)$. Therefore, B can be defined by Δ^0_1 comprehension and again has the required properties. □

Notice that the proof of Lemma 2.4.1 relies on non uniform information: whether or not a separator element $b \in P$ exists. Therefore arbitrarily many applications of Lemma 2.4.1 may not be available in RCA_0. We do not know if it is possible to prove Lemma 2.4.1 uniformly. We point out that in Lemma 2.4.4 we prove that $I\Sigma^0_2$ is required to be able to iterate Lemma 2.4.1 up to an arbitrary $n \in \mathbb{N}$.

Thanks to Lemma 2.4.1 we can prove that each n-DB_p is provable in RCA_0.

Theorem 2.4.2. *For each $n > 0$, $RCA_0 \vdash n$-DB_p.*

Proof. Let $(P, \preceq)$ be a poset. Let $x_0 \in P$ and let $\{\unlhd_0, \dots, \unlhd_{n-1}\}$ be a set of linearization of $(P \setminus \{x_0\}, \preceq)$ witnessing that $\dim(P \setminus \{x_0\}, \preceq) = n$.

Let $I = \{x : x \in P \wedge x \prec x_0\}$ and $F = \{x : x \in P \wedge x_0 \prec x\}$ which exist in RCA_0 and are respectively downward and upward closed in P. Notice that by transitivity $I \prec F$ and since $\prec \subseteq \lhd_0$ then $I \lhd_0 F$. Define two linearizations $\unlhd^0_0, \unlhd^1_0$ of $(P, \preceq)$ starting from $\unlhd_0$ by

$$I \unlhd^0_0 \{x_0\} \unlhd^0_0 P \setminus (\{x_0\} \cup I)$$

$$P \setminus (\{x_0\} \cup F) \unlhd^1_0 \{x_0\} \unlhd^1_0 F$$

where each segment is ordered by $\unlhd_0$. Notice that RCA_0 proves the existence of each segment and (using I, F, B and $\unlhd_0$ as parameters) of the linear orders $\unlhd^0_0$ and $\unlhd^1_0$. It is clear that $\unlhd^0_0$ and $\unlhd^1_0$ are linearizations of $\preceq$. Notice that if $y \unlhd_0 z$ then at least one of $y \unlhd^0_0 z$ and $y \unlhd^1_0 z$ holds.

Next we want to extend each $\unlhd_i$ for $0 < i < n$ to a linearization of $(P, \preceq)$. To this end, for each $0 < i < n$, we apply Lemma 2.4.1 to obtain a $\unlhd_i$-initial interval $B_i \subseteq X \setminus \{x_0\}$ such that $I \subseteq B_i$ and $B_i \cap F = \emptyset$. Then we define the linearization $\unlhd^*_i$ of $(P, \preceq)$ which extends $\unlhd_i$ by

$$B_i \unlhd^*_i \{x_0\} \unlhd^*_i P \setminus (\{x_0\} \cup B_i)$$

where each segment is ordered by $\unlhd_i$.

Finally we claim that the set $\{\unlhd^0_0, \unlhd^1_0, \unlhd^*_1, \dots, \unlhd^*_{n-1}\}$ realizes $(P, \preceq)$. We need to prove that if $z \npreceq y$ then either $y \unlhd^0_0 z$ or $y \unlhd^1_0 z$ or $y \unlhd^*_i z$ for some $0 < i < n$. If $y = x_0$ then $z \notin I$ and $y \unlhd^0_0 z$, while if $z = x_0$ then $y \notin F$ and $y \unlhd^1_0 z$. If neither y nor z are x_0 we have $y \unlhd_i z$ for some $i < n$. If $i = 0$ we already noticed that at least one of $y \trianglelefteq_0^0 z$ and $y \trianglelefteq_0^1 z$ holds. If $i > 0$ then, since $z \in B_i$ and $y \notin B_i$ cannot hold, we have $y \trianglelefteq_i^* z$.

Hence $\dim(P, \preceq) \leqslant \dim(P \setminus \{x_0\}, <) + 1$. □

Statement (3) of Theorem 2.1.3 is a stronger version (equivalent to WKL_0) of Lemma 2.4.1 and was proved in [FM14]. It leads to a proof of DB_p in WKL_0.

Theorem 2.4.3. $\mathsf{WKL}_0 \vdash \mathsf{DB}_p$.

Proof. Let $(P, \preceq)$ be a poset and let $x_0 \in P$. Let $\{\trianglelefteq_0, \dots, \trianglelefteq_{n-1}\}$ be a set of linearizations showing that $\dim(P \setminus \{x_0\}, \preceq) = n$. We need to find a set of $n+1$ linearizations realizing $(P, \preceq)$.

Let $I = \{x : x \in P \wedge x \prec x_0\}$ and $F = \{x : x \in P \wedge x_0 \prec x\}$ which exist in RCA_0. First, we proceed as in Theorem 2.4.2: we define two linearizations $\trianglelefteq_0^0, \trianglelefteq_0^1$ of $(P, \preceq)$ starting from $\trianglelefteq_0$:

$$I \trianglelefteq_0^0 \{x_0\} \trianglelefteq_0^0 P \setminus (\{x_0\} \cup I)$$

$$P \setminus (\{x_0\} \cup F) \trianglelefteq_0^1 \{x_0\} \trianglelefteq_0^1 F$$

where each segment is ordered by $\trianglelefteq_0$. As before, it is clear that $\trianglelefteq_0^0$ and $\trianglelefteq_0^1$ are linearizations of $\preceq$ and that if $y \trianglelefteq_0 z$ then at least one of $y \trianglelefteq_0^0 z$ and $y \trianglelefteq_0^1 z$ holds.

Next, we want to find uniformly for each $0 < i < n$ a separator set B_i for I and F with respect to each linear order $\trianglelefteq_i$: we use Theorem 2.1.3 instead of Lemma 2.4.1. We start by collecting the linearizations $\trianglelefteq_1, \dots, \trianglelefteq_{n-1}$ in one poset $(X, \preceq_X)$. The domain of X is the disjoint union of $n-1$ copies of $P \setminus \{x_0\}$ while the relation $\preceq_X$ is the disjoint union of the linearizations $\trianglelefteq_i$ for $0 < i < n$. When seen as subsets of the i-th copy of $P \setminus \{x_0\}$ in X, we call the sets I and F respectively I_i and F_i. The disjoint unions $\bigsqcup_{i=1}^{n-1} I_i$ and $\bigsqcup_{i=1}^{n-1} F_i$ satisfy the assumption of Theorem 2.1.3. Hence there exists a separator set B for $X, \bigsqcup_{i=1}^{n-1} I_i, \bigsqcup_{i=1}^{n-1} F_i$. Then for each $0 < i < n$, we get a separator set B_i for P, I_i, F_i by intersecting B with the i-th copy of $P \setminus \{x_0\}$ in X. We define the linearization $\trianglelefteq_i^*$ of $(P, \preceq)$ which extend $\trianglelefteq_i$ as in Theorem 2.4.2:

$$B_i \trianglelefteq_i^* \{x_0\} \trianglelefteq_i^* P \setminus (\{x_0\} \cup B_i)$$

where each segment is ordered by $\trianglelefteq_i$.

The proof that the set of linearizations $\{\trianglelefteq_0^0, \trianglelefteq_0^1, \trianglelefteq_1^*, \dots, \trianglelefteq_{n-1}^*\}$ realizes $(P, \preceq)$ is the same as in Theorem 2.4.2. □

Notice that in Theorem 2.4.3 we needed WKL_0 only to prove uniformly the existence of a separator set for I and F in every linearization: this step allows us to add back the point x_0 to each linearization in the right place.

Theorem 2.1.3 shows that, for a generic partial order, being able to produce the separator set is equivalent to WKL_0. However, in our case, $\mathsf{I}\Sigma^0_2$ suffices.

Lemma 2.4.4 (RCA_0). *The following are equivalent:*

(1) $\mathsf{I}\Sigma^0_2$,

(2) for each n*, if* $(P_j, \trianglelefteq_j)_{j<n}$ *is a finite sequence of linear orders and for each* $j < n$*,* $I_j, F_j \subseteq P_j$ *are such that* $I_j \lhd_j F_j$*, then the set of* $j < n$ *for which* P_j*,* I_j *and* F_j *have a separator element exists.*

Proof. (1) implies (2) because the existence of X is an instance of bounded Σ^0_2 comprehension which is equivalent to $\mathsf{I}\Sigma^0_2$ (Theorem 1.1.10).

For the converse, we show that (2) implies bounded Σ^0_2 comprehension. Let $\varphi(j)$ be a Σ^0_2 formula of the form $\exists x \forall y\, \psi(j, x, y)$. For each n, we need to prove that the set $\{j < n : \varphi(j)\}$ exists. We aim to construct a finite sequence of n linear orders $(P_j, \trianglelefteq_j)_{j<n}$, each with subsets I_j, F_j to be separated. We perform the construction to satisfy the following: for each $j < n$, $\varphi(j)$ holds if and only if the linear order $(P_j, \trianglelefteq_j)$ has a separator element. Then it is clear that the set X of (2) is the set $\{j < n : \varphi(j)\}$.

Fix $j < n$. The linear order $(P_j, \trianglelefteq_j)$ has domain $\mathbb{N}$, $I_j = \{m \in \mathbb{N} : m \equiv 0 \mod 3\}$ and $F_j = \{m \in \mathbb{N} : m \equiv 1 \mod 3\}$. We stipulate that $\forall l \in I_j\, \forall m \in F_j\, (l \lhd_j m)$, $\forall l, m \in I_j\, (l \lhd_j m \leftrightarrow l < m)$ and $\forall l, m \in F_j\, (l \lhd_j m \leftrightarrow m < l)$ (where $<$ denotes the standard ordering of $\mathbb{N}$). Since I_j has no maximum and F_j has no minimum, a separator element must belong to the set $Z_j = \{m \in \mathbb{N} : m \equiv 2 \mod 3\}$. We also stipulate that $\forall l \in Z_j\, \forall m \in F_j\, (l \lhd_j m)$ and $\forall l, m \in Z_j\, (l \lhd_j m \leftrightarrow l < m)$

We design a strategy to define the comparabilities of the elements of I_j and Z_j in such a way to ensure that $\varphi(j)$ holds if and only if there is a separator element for P_j, I_j, F_j. In other words, as long as x appears to witness $\varphi(j)$, the strategy keeps $3x + 2$ above all elements of I_j. We proceed by stages: at stage s we stipulate (at least) the comparabilities between each element of Z_j and $3s \in I_j$. The strategy keeps parameters x^s_j for all $j < n$ to mark the current possible existential witness for $\varphi(j)$. We start by setting $x^0_j = 0$ for all $j < n$.

Stage s. For each $j < n$ if $\forall y \leqslant s\, \psi(j, x^s_j, y)$ holds, then we stipulate that $\forall m \geqslant x_j\, (3s \lhd_j 3m + 2)$ and let $x^{s+1}_j = x^s_j$. On the other hand, if $\exists y \leqslant s \neg\psi(j, x^s_j, y)$ holds, then we stipulate that $\forall t \geqslant s\, (3x^s_j + 2 \lhd_j 3t)$ and $\forall m > x^s_j (3s \lhd_j 3m + 2)$. In this second case, we let $x^{s+1}_j = x^s_j + 1$.

This completes the construction of the linear orders $(P_j, \trianglelefteq_j)_{j<n}$.

We are left to prove that for $j < n$, $\varphi(j)$ holds if and only if the linear order $(P_j, \trianglelefteq_j)$ has a separator element. For the forward direction, fix $j < n$ such that $\varphi(j)$ holds. Let $\bar{x}$ be such that $\forall y\,\psi(j,\overline{x},y)$ holds. By construction for each s we stipulated that $3s \lhd_j 3\overline{x}+2$. We conclude that $\overline{x}$ is a separator element for P_j, I_j, F_j.

Conversely, suppose that for $j<n$ $\neg\varphi(j)$ holds. Since $\forall x\,\exists y \neg\psi(j,x,y)$ holds, by $\mathsf{I}\Sigma^0_1$ we have that $\forall x\,\exists s\,(x=x^s_j)$. Fix z large enough so that $\exists y \leqslant z \neg\psi(j,x,y)$ and $\exists s \leqslant z\,(x=x^s_j)$. Then $3x+2 \lhd_j 3z$. Since x was arbitrary, there is no separator element for P_j, I_j, F_j. □

Lemma 2.4.4 implies that assuming $\mathsf{I}\Sigma^0_2$, Lemma 2.4.1 can be applied for an arbitrary finite number of times. This is because, given a finite sequence of linear orders and of sets to be separated, $\mathsf{I}\Sigma^0_2$ is able to recognize when there is a separator element and when not (which is the non computable information we need). Once we know this, we can use the proof of Lemma 2.4.1 to produce a separator set for each linear order.

The existence of a separator set for each linear order of a finite sequence of linear orders follows easily from Theorem 2.1.3 and so is provable in WKL_0. To see this, we just collect all the linear orders of the sequence and all the sets to be separated in a disjoint union as in the proof of Theorem 2.4.3. Since WKL_0 and $\mathsf{I}\Sigma^0_2$ are incomparable, this statement cannot imply neither $\mathsf{I}\Sigma^0_2$ nor WKL_0.

Theorem 2.4.5. $\mathsf{I}\Sigma^0_2 \vdash \mathsf{DB}_p$.

Proof. The proof is essentially the same as in Theorem 2.4.3. Let $(P,\preceq)$ be a poset and let $x_0 \in P$. Let $\{\unlhd_0,\dots,\unlhd_{n-1}\}$ be a set of linearizations showing that $\dim(P\setminus\{x_0\},\preceq)=n$. We need to find a set of $n+1$ linearizations which realizes $(P,\preceq)$.

Let $I=\{x : x\in P \wedge x \prec x_0\}$ and $F=\{x : x \in P \wedge x_0 \prec x\}$ which exist in RCA_0. First, we define two linearizations $\unlhd^0_0, \unlhd^1_0$ of $(P,\preceq)$ starting from $\unlhd_0$ as in the proof of Theorem 2.4.2.

Next, by Lemma 2.4.4 the set X of $0<j<n$ for which the j-th linearization has a separator element for I and F exists. Using X we uniformly define, for each $0<j<n$ a separator set for P, I, F with respect to $\unlhd_j$.

When $j \notin X$ we let $B_j=\{x\in P\setminus\{x_0\} : \exists i \in I\,(x \unlhd_j i\}$. Since there is no separator element for P, I, F with respect to $\unlhd_j$ then $x \in B_j$ if and only if $\forall f \in F(x \lhd f)$ and B_j exists by Δ^0_1 comprehension.

Now we consider the elements of X: by $\mathcal{B}\Pi^0_1$ there exists m such that $\forall j \in X\,\exists b<m\,\forall i \in I\,\forall f \in F\,(i \unlhd_j b \unlhd_j f)$. By bounded Π^0_1 comprehension there exists

$$Y=\{(j,b) : j \in X \wedge b<m \wedge \forall i \in I\,\forall f \in F\,(i \unlhd_j b \unlhd_j f)\}.$$

For each $j \in X$ let b_j be the least b such that $(j,b)\in Y$, so that b_j is a separator element for $(P\setminus\{x_0\},\unlhd_j), I, F$. If $b_j \in F$ then let $B_j=\{x\in P : x \lhd_j b_j\}$, otherwise let $B_j=\{x \in P : x \unlhd_j b_j\}$.

Using B_j, we uniformly define the linearizations $\trianglelefteq_j^*$ of $(P, \preceq)$ which extend $\trianglelefteq_j$ as in the proof of Theorem 2.4.3. Finally, $\{\trianglelefteq_0^0, \trianglelefteq_0^1, \trianglelefteq_1^*, \ldots, \trianglelefteq_{n-1}^*\}$ realizes $(P, \preceq)$ is proved again as in the proof of Theorem 2.4.2. $\square$

Theorems 2.4.3 and 2.4.5 together imply that $\mathsf{DB_p}$ is provable from the disjunction $\mathsf{WKL_0} \vee \mathsf{I}\Sigma_2^0$. Results equivalent to this disjunction do exist, but are rare, in the literature. The only examples known are in [Bel15; FSY93]. In [SY21] a weakening of weak König's lemma is shown to lie strictly between $\mathsf{RCA_0}$ and $\mathsf{WKL_0} \vee \mathsf{I}\Sigma_2^0$.

Even though we do not have a complete reversal of $\mathsf{DB_p}$, we deal with some special cases provable in $\mathsf{RCA_0}$. It is clear from the proofs of Theorems 2.4.3 and 2.4.5 that to prove $\mathsf{DB_p}$ it suffices to produce a separator set for an arbitrarily long finite sequence of linear orders. The following two lemmas are sufficient and deal with the stronger case of infinite sequences of linear orders.

Lemma 2.4.6 ($\mathsf{RCA_0}$)**.** *Let $(P, \preceq)$ be a poset and let $I, F \subseteq P$ be such that $I \prec F$. Suppose there exists a separator element with respect to $\preceq$ and $(P, \trianglelefteq_j)_{j \in \mathbb{N}}$ is a sequence of linearizations of $(P, \preceq)$. Then for every j there exists a separator set with respect to $\trianglelefteq_j$.*

Proof. Each $\trianglelefteq_j$ extends $\preceq$ which implies that for each $i \in I$ and each $f \in F$, $i \trianglelefteq_j b \trianglelefteq_j f$. Therefore if $b \in F$ then for each $j \in \mathbb{N}$ let $B = \{x \in P : x \lhd_j b\}$, otherwise let $B = \{x \in P : x \trianglelefteq_j b\}$. B_j is a separator set with respect to $\trianglelefteq_j$. $\square$

Lemma 2.4.7 ($\mathsf{RCA_0}$)**.** *Let $(P, \preceq)$ be a poset and let $I, F \subseteq P$ be such that for each $i \in I$ and each $f \in F$, $f \not\prec i$. Suppose that for each $x \in P \setminus (I \cup F)$ either there exists $i \in I$ such that $x \preceq i$ or there exists $f \in F$ such that $f \prec x$ and that $(P, \trianglelefteq_j)_{j \in \mathbb{N}}$ is a sequence of linearizations of $(P, \preceq)$. Then for every j there exists a separator set with respect to $\trianglelefteq_j$.*

Proof. Each $\trianglelefteq_j$ extends $\preceq$ which implies that there is no $x \in P$ such that for each $i \in I$ and each $f \in F$, $i \trianglelefteq_j x \trianglelefteq_j f$. We let $B_j = \{x \in P \setminus \{x_0\} : \exists i \in I (x \trianglelefteq_j i\}$. Since there is no separator element for P, I, F with respect to $\trianglelefteq_j$ then $x \in B_j$ if and only if $\forall f \in F (x \lhd f)$ and B_j exists by Δ_1^0 comprehension. $\square$

In view of the previous lemmas, we can restrict to the case where there is no separator element already in the poset $(P, \preceq)$ but there are possible candidates for the separator element that arise when we linearize. Clearly not every $b \in P$ is a candidate to be a separator element in a linearization: $b \in P$ is a suitable candidate to be a separator element in a linearization if and only if $\forall i \in I (b \not\preceq i)$ and $\forall f \in F (f \not\preceq b)$. We call C the set of suitable candidates to be a separator element. In our proofs of $\mathsf{DB_p}$, C is $\{b \in P : b | x_0\}$.

Lemma 2.4.8 ($\mathsf{RCA_0}$)**.** *If the set of suitable candidates C is finite then for any finite sequence $(P, \trianglelefteq_j)_{j<n}$ of linearizations of $(P, \preceq)$ there exists a separator set.*

Proof. Since C is a finite set, the formula $\exists b \in C \, \forall i \in I_j \, \forall f \in F_j \, (i \trianglelefteq_j b \trianglelefteq_j f)$ is of the form $(\exists b < t)\varphi(b)$ where $\varphi(b)$ is Π^0_1. Then $\mathcal{B}\Sigma^0_1$ proves that this formula is equivalent to a Π^0_1 formula (see [DM22, Theorem 6.1.2]). It follows by bounded Σ^0_1 comprehension that the set

$$X = \{j < n : \exists b \in C \, \forall i \in I_j \, \forall f \in F_j \, (i \trianglelefteq_j b \trianglelefteq_j f)\}$$

exists in RCA_0. From here we continue as in the proof of Theorem 2.4.5. □

The only case left is when the set C of suitable candidates is infinite: in our proofs of DB_p when the set of elements incomparable with x_0 is infinite.

3 STRONG GRAPH INDIVISIBILITY

This chapter is joint work with Damir Dzhafarov and Reed Solomon.

Versions of Ramsey's theorem have been a consistent source of important principles in computable combinatorics and reverse mathematics. In recent years, Weihrauch reducibility and its variants have given new tools and perspectives to study uniformity questions in these contexts and to make fine distinction between proof methods. The majority of this work considers subsets of size $n \geqslant 2$ with an emphasis on $n = 2$. However, even the $n = 1$ cases, or pigeonhole principles, often have subtle connections to uniformity and induction.

The canonical case is Ramsey's theorem for singletons: every coloring $c \colon \omega \to k$ has an infinite monochromatic set. From the perspective of reverse mathematics, for fixed k, RT^1_k is provable in RCA_0, while it was showed in [Hir87] that the full result RT^1 is equivalent to the induction scheme $\mathsf{B}\Sigma^0_2$ (see Definition 1.1.8). On the Weihrauch side, non reductions were found for a variety of reducibilities in [Dor+16; BR17; HJ16; Pat16; Dzh+17].

Adding some structure, in [CHM09] it was introduced a tree version of RT^1: every coloring $c : 2^{<\omega} \to k$ has a monochromatic subset H isomorphic to $2^{<\omega}$ as a partial order. In reverse mathematics, this principle is denoted TT^1 and was shown to lie strictly between the induction schemes $\mathsf{B}\Sigma^0_2$ and $\mathsf{I}\Sigma^0_2$ in [CGM10] and in [Cho+20]. Interestingly, Kołodziejczyck has a proof (reproduced in [DSV25]) using results from [Cho+20] and [CGM10] to show that TT^1, unlike RT^1, is not equivalent to any arithmetical statement. A detailed account of the relationship between the singleton versions of Ramsey's theorem and the tree theorem in the Weihrauch degrees including a Weihrauch version of Kołodziejczyck's result is given in [DSV25].

More generally, a relational structure M (which we identify with its domain for simplicity) is *indivisible* if for every finite coloring $c : M \to k$ of the domain of M, there is a monochromatic subset H of M such that the induced substructure on H is isomorphic to M. In this setting, in [Gil25] is considered a variety of Weihrauch problems related to indivisible structures such as the dense linear order $(\mathbb{Q}, \leqslant)$ and the random graph $\mathcal{R}$.

There is a significant difference between, on one hand, Ramsey's theorem for singletons and the indivisibility of $\mathcal{R}$, and on the other hand, TT^1 and the indivisibility of $(\mathbb{Q}, \leqslant)$. For a coloring $c \colon \omega \to k$, there is a color i such that $c^{-1}(i)$ is an infinite monochromatic set. That is, one of the colors is the desired subset for RT^1. Similarly, for $c : \mathcal{R} \to k$, one of the colors is an isomorphic copy of $\mathcal{R}$, a fact first pointed out in [Hen71]. However, there are colorings of $2^{<\omega}$ and $\mathbb{Q}$ such that no full color is isomorphic to $2^{<\omega}$ or $\mathbb{Q}$ as a partial or linear order respectively.

Our concern here is with the stronger property that requires the full set of some color to yield an isomorphic substructure. In the combinatorics literature, colorings are often replaced with partitions and a relational structure M is said to have the *pigeonhole property* if for any finite partition $M = X_0 \sqcup \cdots \sqcup X_{k-1}$, there is an i such that the induced substructure on X_i is isomorphic to M. Setting aside induction issues, this definition is often equivalently stated with partitions into two pieces, i.e. $M = X_0 \sqcup X_1$.

Unfortunately, the term pigeonhole property is frequently used in computability theory and reverse mathematics for a Ramsey style property that only requires a subset of a color to induce an isomorphic structure. Therefore, to avoid terminological conflict, we say M is *strongly indivisible* if for every partition $M = X_0 \sqcup X_1$, the induced substructure on X_0 or X_1 is isomorphic to M.

Strong indivisibility (under the name pigeonhole property) appears to have been introduced in [Cam97] where it was proved that there are exactly three strongly indivisible countable graphs: the complete graph K_ω, the completely disconnected graph $\overline{K}_\omega$ and the random graph $\mathcal{R}$. There are similar classifications of the strongly indivisible tournaments, posets and linear orders in [BCD00] as well as a study of the connection between Fraïsse limits and strong indivisibility in [BD99].

Our goal is to examine Cameron's classification of the strongly indivisible countable graphs from the point of view of reverse mathematics and computable combinatorics.

In Section 3.1, we give a classical proof of the classification, showing it can be done in ACA_0 and pointing out where it uses arithmetical comprehension and induction axioms beyond $\mathsf{I}\Sigma^0_1$.

In Section 3.2, we show that the classification is effective up to computable presentation. That is, if G is a computable graph not isomorphic to K_ω, $\overline{K}_\omega$ or $\mathcal{R}$, then there is a computable copy H of G and a computable partition $H = X_0 \sqcup X_1$ such that neither the induced graph on X_0 nor the induced graph on X_1 is even classically isomorphic to G. We show the move from G to H is necessary to get this strong result by constructing a computable graph G that is not isomorphic to K_ω, $\overline{K}_\omega$ or $\mathcal{R}$, but for every computable partition $G = X_0 \sqcup X_1$, the induced subgraph on at least one of X_0 or X_1 is classically isomorphic to G.

In Section 3.3, we provide a partial result towards showing the classification theorem holds in the ω-model REC. If G is a computable graph that is not isomorphic to K_ω, $\overline{K}_\omega$ or $\mathcal{R}$ and for which the set of vertices of finite degree is c.e., then we show there is a computable partition $G = X_0 \sqcup X_1$ such that neither X_0 nor X_1 is computably isomorphic to

G. However, the full question of whether REC satisfies the classification theorem remains open.

Finally, in Section 3.4, we return to Cameron's original proof of the classification. The classical proof of the classification theorem applies $L\Sigma_2^0$ (see Definition 1.1.8) to a graph $G \not\cong \mathcal{R}$ to obtain the smallest size counterexample in G to the extension property that characterizes the random graph. We show that the existence of a counterexample to the extension property of minimal size in every non random graph is equivalent to the full induction scheme $L\Sigma_2^0$.

3.1 THE CLASSICAL PROOF

We introduced the basics of graph theory in Section 1.2. We also noticed that in the language of second order arithmetic we can characterize the random graph $\mathcal{R}$ with a unique sentence that we called $\varphi_{\mathcal{R}}$.

In this section, we give Cameron's proof classifying the countable strongly indivisible graphs with an eye towards formalizing it in reverse mathematics. In the context of a model of a subsystem of second order arithmetic, recall that we let $\mathbb{N}$ denote the first order part of the model, while ω denotes the standard natural numbers.

We begin by showing that RCA_0 proves that both K_ω and $\overline{K}_\omega$ are strongly indivisible.

Proposition 3.1.1 (RCA_0). *K_ω and $\overline{K}_\omega$ are strongly indivisible.*

Proof. Fix a partition of the vertices $\mathbb{N} = X_0 \sqcup X_1$. At least one of X_0 and X_1 is infinite, so the corresponding subgraph is isomorphic to K_ω or to $\overline{K}_\omega$. □

This symmetry between K_ω and $\overline{K}_\omega$ with respect to strong indivisibility extends more generally. For a graph $G = (V, E)$, recall that $\overline{G}$ denotes the graph obtained by swapping the edges and non edges, except along the diagonal. Formally, $\overline{G} = (V, \overline{E})$ with $\overline{E} = \{\langle m, n\rangle \notin E : m \neq n\}$. Recall that we frequently abuse notation by equating a graph with its domain.

Proposition 3.1.2 (RCA_0). *G is strongly indivisible if and only if $\overline{G}$ is strongly indivisible.*

Proof. Let $V = X_0 \sqcup X_1$ be a partition of the vertices, and let H_i and $\overline{H}_i$ denote the corresponding subgraphs in G and $\overline{G}$. Because a graph isomorphism preserves both edges and non edges, a bijection $f : X_i \to V$ is an isomorphism from H_i to G if and only if it is an isomorphism from $\overline{H}_i$ to $\overline{G}$. □

We show some important properties of the random graph.

Proposition 3.1.3 (RCA_0)**.** *Let $\mathcal{R}$ be a random graph.*

(1) Let A and B be disjoint finite sets of vertices in $\mathcal{R}$. The subgraph $G_{A,B}$ on $V_{A,B} = \{x \in \mathcal{R} \setminus (A \cup B) : (\forall a \in A) E(x, a) \wedge (\forall b \in B) \neg E(x, b)\}$ is a random graph.

(2) $\mathcal{R}$ is strongly indivisible.

Proof. For (1), to show $G_{A,B}$ is a random graph, consider disjoint finite sets $C, D \subseteq V_{A,B}$. Since $\mathcal{R}$ is random, there is a node x such that $E(x, y)$ for all $y \in A \cup C$ and $\neg E(x, z)$ for all $z \in B \cup D$. It follows that $x \in V_{A,B}$ and that x witnesses the extension axiom for the finite sets C and D in $G_{A,B}$.

For (2), fix a partition $\mathcal{R} = X_0 \sqcup X_1$. Suppose for a contradiction that neither of the induced subgraphs are random. For $i < 2$, fix disjoint finite sets $A_i, B_i \subseteq X_i$ for which the extension axiom fails in the induced subgraph. Since $\mathcal{R}$ is random, there is an x such that $E(x, a)$ for all $a \in A_0 \cup A_1$ and $\neg E(x, b)$ for all $b \in B_0 \cup B_1$. Let $j < 2$ be such that $x \in X_j$ and notice that x witnesses the extension property for the pair A_j, B_j in X_j, contrary to the assumption. □

We turn to showing K_ω, $\overline{K}_\omega$ and $\mathcal{R}$ are the only strongly indivisible countable graphs. Since every nontrivial partition of a finite graph witnesses that it is not strongly indivisible, we only need to consider infinite graphs.

Theorem 3.1.4 (Cameron [Cam97])**.** *If G is a countable strongly indivisible graph, then G is isomorphic to K_ω, $\overline{K}_\omega$ or $\mathcal{R}$.*

Proof. Assume that G is an infinite graph that is not isomorphic to K_ω, $\overline{K}_\omega$ or $\mathcal{R}$.

Case 1. Suppose G has isolated vertices. Let $X_0 = \{x \in G : x \text{ is isolated}\}$ and $X_1 = G \setminus X_0$. The induced subgraph on X_0 is $\overline{K}_{|X_0|}$, which is not isomorphic to G by assumption. If x were an isolated vertex in the induced subgraph X_1, then, in fact, x would be an isolated vertex in G, and hence $x \in X_0$. Therefore, X_1 has no isolated vertices and so is not isomorphic to G.

Case 2. Suppose G has universal vertices. This case follows by letting X_0 be the set of universal vertices and reasoning as in Case 1.

Case 3. Suppose G has neither isolated nor universal vertices. Since G is not a random graph, let n be least such that there are disjoint sets A and B for which $|A| + |B| = n$ and the extension axiom $\varphi_{\mathcal{R}}$ fails for A and B.

We claim that $n \geqslant 2$. We cannot have $n = 0$ because G is nonempty, so suppose $n = 1$. If $A = \{a\}$ and $B = \emptyset$, then the failure of $\varphi_{\mathcal{R}}$ implies a is an isolated vertex, contrary to our case assumption. On the other hand, if $A = \emptyset$ and $B = \{b\}$, then the failure of the extension axiom for A, B means b is a universal vertex, also contrary to our case assumption.

Since $n \geqslant 2$, we can partition $A \cup B = U_0 \sqcup U_1$ into nonempty sets U_0 and U_1. We say a vertex x is *not correctly joined to* U_i if there is an $a \in U_i \cap A$ such that $\neg E(x, a)$ or there is a $b \in B \cap U_i$ such that $E(x, b)$. Since the extension axiom does not hold for A and B, every vertex x is not correctly joined to at least one of U_0 or U_1.

Let $X_0 = U_0 \cup \{x \in G : x \notin U_1 \wedge x \text{ is not correctly joined to } U_0\}$ and let $X_1 = G \setminus X_0$. Note that $G = X_0 \sqcup X_1$, $U_1 \subseteq X_1$, and every node in X_1 is not correctly joined to U_1. By construction, X_0 fails to satisfy the extension axiom with $A \cap U_0$ and $B \cap U_0$, while X_1 fails to satisfy the extension axiom with $A \cap U_1$ and $B \cap U_1$. Since U_0 and U_1 are nonempty, it follows that X_0 and X_1 fail to satisfy instances of the extension axiom for which the sum of the sizes of the witnessing sets is strictly less than n. Therefore, since n was chosen least for G, neither X_0 nor X_1 is isomorphic to G. □

On its face, this proof uses axioms outside of RCA_0 in three places. In Cases 1 and 2, it uses arithmetical comprehension to form the sets of isolated and universal vertices. In Case 3, it uses the existence of the least $n \in \mathbb{N}$ such that

$$\exists \text{disjoint } A, B \left(|A| + |B| = n \wedge \forall x \left[(\exists a \in A) \neg E(x, a) \vee (\exists b \in B) E(x, b)\right]\right).$$

Since the least number axiom schema $\mathsf{L}\Sigma^0_2$ is equivalent to $\mathsf{I}\Sigma^0_2$, which holds in ACA_0, Theorem 3.1.4 is provable in ACA_0.

We prove that the existence of the set of isolated or universal vertices is equivalent to ACA_0. Similarly, we also prove that $\mathsf{L}\Sigma^0_2$ is equivalent to the least number principle restricted to formulas of the form above. We give the equivalence with ACA_0 here because it is short, but delay the $\mathsf{L}\Sigma^0_2$ equivalence until Section 3.4. These equivalences do not give us lower bounds on the strength of Theorem 3.1.4, but they do tell us that the proof above cannot be carried out in a system weaker than ACA_0.

Proposition 3.1.5 (RCA_0)**.** *The following are equivalent.*

(1) ACA_0.

(2) for every graph, the set of universal nodes exists.

(3) for every graph, the set of isolated nodes exists.

Proof. The implications from (1) to (2) and from (1) to (3) hold because these sets are arithmetically definable.

For the implication from (2) to (3), fix a graph G. The set of isolated nodes in G is the same as the set of universal nodes in $\overline{G}$, which exists by (2).

For the implication from (3) to (1), let $f\colon \mathbb{N} \to \mathbb{N}$ be an arbitrary 1-1 function. It suffices to show the range of f exists (see Theorem 1.1.6). Define G with $V = \mathbb{N}$ and a symmetric edge between x and y if and only if $x = 2n$, $y = 2m + 1$ and $f(n) = m$. The range of f is definable from the set of isolated nodes in G because an odd vertex $2m + 1$ is isolated if and only if m is not in the range of f. □

3.2 EFFECTIVENESS UP TO PRESENTATION

Our motivating question is whether the classification of strongly indivisible graphs is provable in RCA_0, and in particular, whether it holds in the ω-model REC. In the previous section, we showed one direction holds in RCA_0, which translates to REC as follows.

Theorem 3.2.1. *Let G be a computable graph that is isomorphic to K_ω, $\overline{K}_\omega$ or $\mathcal{R}$. For every computable partition $G = X_0 \sqcup X_1$, either X_0 or X_1 is computably isomorphic to G.*

Because the graphs K_ω, $\overline{K}_\omega$ and $\mathcal{R}$ are computably categorical, it does not matter whether we use "isomorphic" or "computably isomorphic" in the statement of Theorem 3.2.1. However, in general, it is possible for the effectiveness properties to vary across computable presentations of a graph. We show that Theorem 3.1.4 is effective up to computable presentation in a strong form in which we consider the classical isomorphism types of the partition pieces.

Theorem 3.2.2. *Let G be a computable graph that is not isomorphic to K_ω, $\overline{K}_\omega$ or $\mathcal{R}$. There is a computable presentation H of G and a computable partition $H = X_0 \sqcup X_1$ such that neither X_0 nor X_1 is classically isomorphic to G.*

To prove Theorem 3.2.2, we use the following theorem and corollary to mimic the classical proof of Theorem 3.1.4.

Theorem 3.2.3. *Every computable graph G has a computable copy in which the set of isolated vertices is computable.*

Corollary 3.2.4. *Every computable graph G has a computable copy in which the set of universal vertices is computable.*

Corollary 3.2.4 follows immediately from Theorem 3.2.3 by shifting from G to $\overline{G}$. At the end of the section, we return to a proof of Theorem 3.2.3. For now, we use these results to prove Theorem 3.2.2.

Proof of Theorem 3.2.2. We follow the classical proof of Theorem 3.1.4 given above. In Case 1, when G has isolated vertices, we apply Theorem 3.2.3 to get a computable copy H for which the partition $H = X_0 \sqcup X_1$ is computable, where X_0 is the set of isolated vertices. In Case 2, when G has universal vertices, we apply Corollary 3.2.4 to get a computable copy H for which the partition $H = X_0 \sqcup X_1$ is computable, where X_0 is the set of universal vertices. In either case, the proof that neither X_0 nor X_1 is classically isomorphic to G is the same as in Theorem 3.1.4.

For Case 3, when G has neither isolated nor universal vertices, we run the argument from Theorem 3.1.4 without changing the presentation of G. The least value n exists in the standard natural numbers, and the partition pieces X_0 and X_1 are computable because they are defined with bounded quantifiers. □

Our next goal is to show that the shift of computable presentations in Theorem 3.2.2 is necessary to get a strong effectiveness result that considers the partition pieces up to classical isomorphism.

Let $K^{\infty}_{<\omega}$ denote the graph consisting of infinitely many disjoint copies of K_n for each $n \geqslant 1$. We say that a copy of K_n inside $K^{\infty}_{<\omega}$ is *finished* if it is not a subgraph of a larger K_m inside $K^{\infty}_{<\omega}$.

There is a nice computable copy H of $K^{\infty}_{<\omega}$ for which there is a computable function f such that each vertex x sits in a finished copy of $K_{f(x)}$. There are many computable partitions $H = X_0 \sqcup X_1$ such that neither X_0 nor X_1 is classically isomorphic to $K^{\infty}_{<\omega}$. For example, let X_0 be the set of isolated nodes (i.e. those for which $f(x) = 1$), or more generally, let X_0 be the set of all nodes for which $f(x) = n$ for any fixed n.

However, $K^{\infty}_{<\omega}$ also has computable copies which are less uniformly constructed. In the next theorem, we build a computable $G \cong K^{\infty}_{<\omega}$ such that every computable partition $G = X_0 \sqcup X_1$ has at least one X_i classically isomorphic to $K^{\infty}_{<\omega}$.

The isomorphism type of $K^{\infty}_{<\omega}$ has several properties that make it suitable for this construction. Deleting finitely many vertices does not change its isomorphism type, and neither does adding countably many disjoint copies of finished graphs K_n for each $n \in \omega$. Moreover, a subgraph of K_n is isomorphic to K_m for some $m \leqslant n$. Therefore, if $K^{\infty}_{<\omega} = X_0 \sqcup X_1$, then each finished component of X_i is a copy of K_m for some m. It follows that X_i is isomorphic to $K^{\infty}_{<\omega}$ as long as it contains infinitely many finished copies of K_m for each m, i.e. we do not have to worry about what other finished components X_i contains.

Let Φ_e be the e-th computable function. In the proof of the following theorem, it is convenient to regard each Φ_e as $\{0,1\}$-valued and to use K_0 to denote the empty set.

Theorem 3.2.5. *There is a computable $G \cong K^{\infty}_{<\omega}$ such that for every computable partition $G = X_0 \sqcup X_1$, either X_0 or X_1 is classically isomorphic to $K^{\infty}_{<\omega}$.*

Proof. We build G computably in stages with G_s denoting the graph at the end of stage s. The set of vertices in G_s will be a finite initial segment of ω. We neither add nor delete edges between vertices in G_s after stage s.

Let π_1 denote the projection function onto the first coordinate. As s goes to infinity, the values of $\pi_1(s)$ hit each natural number infinitely often. We use this property to ensure G is isomorphic to $K^{\infty}_{<\omega}$. At the start of stage s, we add $\pi_1(s)+1$ new vertices and put edges between them to form a finished copy of $K_{\pi_1(s)+1}$. This action ensures G has a subgraph isomorphic to $K^{\infty}_{<\omega}$. Therefore, as long as each additional finished component in G has the form K_n for some n, G will be isomorphic to $K^{\infty}_{<\omega}$.

For each index e, we let $X^e_0 = \{x : \Phi_e(x) = 0\}$ and $X^e_1 = \{x : \Phi_e(x) = 1\}$. If Φ_e is total, these sets partition G. We list our requirements as follows.

$$R_e : \text{If } \Phi_e \text{ is total, then } X^e_0 \cong K^{\infty}_{<\omega} \text{ or } X^e_1 \cong K^{\infty}_{<\omega}.$$

To satisfy this requirement, it suffices to ensure that at least one X^e_i contains infinitely many finished copies of K_n for each n. We first describe informally our strategy to meet the requirements R_e. Then we give the formal construction.

The R_e module keeps three parameters: numbers m^e_0 and m^e_1, and a finite set C^e. Each parameter will change during the construction. We typically suppress denoting the stage, but write $m^e_{i,s}$ and C^e_s (and later $C^e_{i,s}$) when we need to explicitly reference the current stage s. The numbers m^e_0 and m^e_1 track the finished graphs K_n we have seen in X_0 and X_1 respectively. For each $n < m^e_i$, we will already have forced a finished copy of $K_{\pi_1(n)+1}$ into X_i with separate copies when $n \neq n'$ and $\pi_1(n) = \pi_1(n')$.

The current goal for R_e is to create a finished copy of $K_{\pi_1(m^e_0)+1}$ in X_0 or $K_{\pi_1(m^e_1)+1}$ in X_1. To meet this goal, add a new element y_0 to G_s with no edges, set $C^e = \{y_0\}$ and let C^e_i empty for $i < 2$. Currently, C^e is a copy of K_1 and each C^e_i is a copy of K_0. If $\Phi_e(y_0)$ halts at a future stage s_0, one of the C^e_i sets becomes a copy of K_1 and the other remains a copy of K_0. If $C^e_i \cong K_{\pi_1(m^e_i)+1}$ for an $i < 2$, then we have met our goal on the X_i side. In this case, set $m^e_i = m^e_i + 1$, empty C^e, leave m^e_{1-i} unchanged, and restart the R_e module with the new parameters.

Otherwise, we add a new element y_1 to G_{s_0}, connect it to y_0, and expand $C^e = \{y_0, y_1\}$ to a copy of K_2. If $\Phi_e(y_1)$ halts at some later stage s_1, one of the C^e_i sets grows by one element. If $C^e_i \cong K_{\pi_1(n_i)+1}$, then we have met our goal on the X_i side, and so we increment m^e_i, empty C^e, and restart the R_e module with the new parameters. If we have not met the goal on either side, add a new vertex y_2 to G_{s_1}, connect it to y_0 and y_1, expand $C^e = \{y_0, y_1, y_2\}$ to a copy of K_3, and repeat the process above.

We cannot cycle through this process infinitely often because when $\Phi_e(y_k)$ halts, we have $|C_0^e| + |C_1^e| = k+1$. Therefore, before $|C^e| = \pi_1(m_0^e) + \pi_1(m_1^e) + 2$, one of the C_i^e sets must reach $|C_i^e| = \pi_1(m_i^e) + 1$, and so satisfies $C_i^e \cong K_{\pi_1(m_i^e)+1}$, meeting our goal on the X_i side.

When we restart the R_e module at a stage s, we empty C^e (i.e. set $C_s^e = \emptyset$) and begin again with C_s^e starting a new connected component. After this stage, we never add vertices to the old component C_{s-1}^e. Therefore, C_{s-1}^e is a finished component in G, each $C_{i,s-1}^e$ is finished in X_i, and so we have created a finished copy of $K_{\pi_1(m_{i,s-1}^e)+1}$ in X_i for the $i<2$ such that $C_{i,s-1}^e \cong K_{\pi_1(m_{i,s-1}^e)+1}$.

Furthermore, each time we restart the R_e module, one of the m_i^e parameters is incremented. Therefore, if Φ_e is total, the values of at least one m_i^e go to infinity, causing $\pi_1(m_i^e)$ to cycle through each number infinitely often. It follows that X_i contains infinitely many finished copies of K_n for each $n \geqslant 1$ and hence is classically isomorphic to $K_{<\omega}^{\infty}$.

The formal construction proceeds as follows. At stage 0, set $G_0 = C^0 = \{0\}$, set $m_0^e = m_1^e = 0$ for all e, and set $C^e = \emptyset$ for $e > 0$.

At stage $s > 0$, let $k = \pi_1(s) + 2 + |\{e < s : \Phi_{e,s}(\max C^e) \text{ halts}\}|$, $X_s = \{x_0, \cdots, x_{k-1}\}$ be the k least unused numbers, and $G_s = G_{s-1} \cup X_s$. Add edges between x_i and x_j for $i \neq j \leqslant \pi_1(s)$ to create a finished copy of $K_{\pi_1(s)+1}$. Set $C^s = \{x_{\pi_1(s)+1}\}$ and leave the remaining parameters for $e \geqslant s$ unchanged.

Consider the indices $e < s$ in order. If $\Phi_{e,s}(\max C^e)$ (namely, the computable function Φ_e on input $\max C^e$ after s steps) does not halt, leave m_i^e and C^e unchanged and go to $e+1$. If $\Phi_{e,s}(\max C^e)$ halts, then check whether $C_i^e = \{z \in C^e : \Phi_{e,s} = i\} \cong K_{\pi_1(m_i^e)+1}$ for some $i < 2$. If not, set $C_s^e = C_{s-1}^e \cup \{x_\ell\}$ where x_ℓ is the least unused number from X_s. Connect x_ℓ to each element of C_{s-1}^e so that $C_s^e \cong K_{|C^e|}$. Leave m_0^e and m_1^e unchanged and go to $e+1$. Finally, if $C_i^e \cong K_{\pi_1(m_i^e)+1}$, then set $C_s^e = \{x_\ell\}$, $m_{i,s}^e = m_{i,s-1}^e + 1$, $m_{1-i,s}^e = m_{1-i,s-1}^e$, and go to $e+1$.

This completes the formal construction. The details of the verification are essentially contained in the informal description above as there is no interaction between the requirements with different indices. □

We end this section with the proof of Theorem 3.2.3.

Proof of Theorem 3.2.3. Fix a computable graph G and assume without loss of generality that the set of vertices is ω. Suppose the set of isolated nodes is not computable, and hence there are infinitely many isolated nodes as well as infinitely many non isolated nodes. Let G_s denote the subgraph on $\{0, \ldots, s\}$.

We build a computable graph H and a Δ_2^0 isomorphism $f\colon G \to H$ in stages such that the isolated nodes in H are exactly the even numbers. At stage s, we define an injection f_s on

G_s and let H_s denote the range of f_s with edge relation defined by $E_{H_s}(n,m)$ if and only if $E_{G_s}(f_s^{-1}(n), f_s^{-1}(m))$. Thus, by definition, f_s will be an isomorphism from G_s to H_s. To make the edge relation on H computable, we ensure that $E_{H_s}(n,m)$ implies $E_{H_t}(n,m)$ for all $t \geqslant s$ such that $n, m \in H_t$.

Again we start by giving the informal idea of the construction. First notice that the domains of the graphs H_s will not necessarily be monotonic. Suppose x is isolated in G_s, so we set $f_s(x) = 2n$ to map x to an even number in H_s. If we discover x is not isolated in G_{s+1} by seeing $E_G(x, s+1)$, then we need to shift $f_{s+1}(x)$ to an odd number. We collect the nodes $x_0 < \cdots < x_\ell$ that were isolated in G_s but are connected to $s+1$, and we map these nodes to the least odd numbers not in H_s. Next, we collect the nodes $a_0 < \cdots < a_j$ (if any) that remain isolated in G_{s+1} and map these elements onto an initial segment of the even numbers. Since the number of isolated nodes has gone down from G_s to G_{s+1}, at least one even number in H_s is no longer in H_{s+1}. However, because G has infinitely many isolated nodes, each even number will eventually be permanently in the range of the f_s maps.

We now give the construction. At stage 0, set $f_0(0) = 0$, noting that 0 is isolated in G_0. At stage $s+1$, we define f_{s+1} as follows.

Case 1: $s+1$ is isolated in G_{s+1}. Let m be the least even number such that $m \notin H_s$. Define $f_{s+1}(x) = f_s(x)$ for $x \leqslant s$ and $f_{s+1}(s+1) = m$.

Case 2: $s+1$ is not isolated in G_{s+1} but is not attached to any nodes which are isolated in G_s. Let k be the least odd number such that $k \notin H_s$. Define $f_{s+1}(x) = f_s(x)$ for $x \leqslant s$ and $f_{s+1}(s+1) = k$.

Case 3: $s+1$ is not isolated in G_{s+1} and it is attached to at least one node which is isolated in G_s. Let $x_0 < \cdots < x_\ell$ denote the nodes which are isolated in G_s but are connected to $s+1$ in G_{s+1}. Let $a_0 < \ldots < a_j$ denote the nodes (if any) which are isolated in G_{s+1}. Let $k_0 < \cdots < k_{\ell+1}$ denote the least odd numbers not in H_s. For $x \leqslant s+1$, define

$$f_{s+1}(x) = \begin{cases} 2i & \text{if } x = a_i \\ k_i & \text{if } x = x_i \\ k_{\ell+1} & \text{if } x = s+1 \\ f_s(x) & \text{otherwise} \end{cases}$$

This completes the construction. It is straightforward to check a number of properties by induction on s. First, each function f_s is injective. Second, H_s consists of the union of an initial segment of the even numbers and an initial segment of the odd numbers. Third, x is isolated in G_s if and only if $f_s(x)$ is even. Therefore, if $n \in H_s$ is even, then $\neg E_{H_s}(n,m)$

for all $m \in H_s$. Fourth, if $m \in H_s$ and $m \notin H_{s+1}$, then m is even. Fifth, if m is odd and $f_s(x) = m$, then $f_t(x) = m$ for all $t \geqslant s$.

Lemma 3.2.6. *For $s < t$ and $m, n \in H_s \cap H_t$, $E_{H_s}(m,n)$ if and only if $E_{H_t}(m,n)$.*

Proof. If m or n is even, then by the third property above, $\neg E_{H_s}(n,m)$ and $\neg E_{H_t}(n,m)$. Therefore, assume m and n are odd. Fix $x, y \in G_s$ with $f_s(x) = m$ and $f_s(y) = n$. By the fifth property, $f_t(x) = m$ and $f_t(y) = n$, and so by definition, $E_{H_s}(n,m)$ and $E_{H_t}(n,m)$ are both equivalent to $E_G(x,y)$. □

Lemma 3.2.7. *For each x, there is a stage s such that $f_t(x) = f_s(x)$ for all $t \geqslant s$.*

Proof. Suppose x is not isolated and let y be the least node such that $E_G(x,y)$. If $y < x$, then x is not isolated in G_x and therefore $f_x(x)$ is odd. If $x < y$, then at stage y the construction acts in Case 3 and the value $f_y(x)$ is odd. In either case, once x is mapped to an odd number, $f_s(x)$ has stabilized.

Suppose x is isolated and so $f_x(x)$ is even. For $s \geqslant x$, $f_{s+1}(x) \neq f_s(x)$ only if a node $y < x$ is isolated in G_s but is connected to $s+1$. In this case, $f_{s+1}(y)$ becomes odd and $f_{s+1}(x) < f_s(x)$ is a smaller even number. This drop can happen at most finitely often before reaching a limit value. □

Lemma 3.2.8. *For each n, there is an $x \in G$ and a stage s such that $f_t(x) = n$ for all $t \geqslant s$.*

Proof. Suppose n is odd. Fix s such that G_s contains at least $(n+1)/2$ non isolated nodes. Since f_s maps the non isolated nodes of G_s onto an initial segment of the odd numbers, there is an $x \in G_s$ such that $f_s(x) = n$. Because n is odd, $f_t(x) = f_s(x) = n$ for all $t \geqslant s$.

Suppose n is even. Let $a_0 < \cdots < a_{n/2}$ be an initial segment of the isolated nodes in G. Fix s such that these nodes form an initial segment of the isolated nodes in G_s. For every $t \geqslant s$, f_t maps these nodes onto an initial segment of the even numbers, and therefore, $f_t(a_{n/2}) = n$ for all $t \geqslant s$. □

Define $H = (\omega, E_H)$ with $E_H(n,m)$ holds if and only if $E_{H_s}(n,m)$ holds for the least s with $n, m \in H_s$. By Lemma 3.2.6, $E_H(n,m)$ holds if and only if $E_{H_s}(n,m)$ holds for some, or equivalently all, s with $n, m \in H_s$. It follows that n is isolated in H if and only if n is even.

By Lemmas 3.2.7 and 3.2.8, the function $f = \lim_s f_s$ is Δ^0_2, total and onto ω. It is injective because each f_s is injective, so $f \colon \omega \to \omega$ is a bijection. To finish the proof, we show that f is an isomorphism between G and H.

Lemma 3.2.9. $f \colon G \to H$ *is an isomorphism.*

Proof. Fix $x, y \in G$ and $s \geqslant \max\{x, y\}$ such that $f(x) = f_s(x)$ and $f(y) = f_s(y)$.

$$E_G(x,y) \Leftrightarrow E_{G_s}(x,y) \Leftrightarrow E_{H_s}(f_s(x), f_s(y)) \Leftrightarrow E_H(f(x), f(y)).$$

The first equivalence follows from $x, y \in G_s$, the second follows from the definition of E_{H_s}, and the third follows because $f(x) = f_s(x)$ and $f(y) = f_s(y)$. □

This completes the proof of Theorem 3.2.3. □

3.3 TOWARDS AN ANALYSIS IN REC

Theorem 3.1.4 holds in REC if and only if for every computable graph G not isomorphic to K_ω, $\overline{K}_\omega$ or $\mathcal{R}$, there is a computable partition $G = X_0 \sqcup X_1$ such that neither X_0 nor X_1 is computably isomorphic to G. While this full statement remains open, we handle a special case in this section.

If G has no isolated or universal vertices, then as noted in the proof of Theorem 3.2.2, there is a computable partition such that neither half is even classically isomorphic to G. Therefore, to study Theorem 3.1.4 in REC, we can restrict our attention to computable graphs that have isolated or universal vertices. Moreover, since isolated nodes in G correspond to universal nodes in $\overline{G}$, we can apply Proposition 3.1.2 to restrict to computable graphs that have isolated nodes. It follows that Theorem 3.1.4 holds in REC if and only if for every computable graph G that has isolated nodes but is not isomorphic to $\overline{K}_\omega$, there is a computable partition $G = X_0 \sqcup X_1$ such that neither X_0 nor X_1 is computably isomorphic to G. We establish this statement under the additional hypothesis that the set of vertices of finite degree is computably enumerable.

Theorem 3.3.1. *Let G be a computable graph that has isolated vertices but is not isomorphic to $\overline{K}_\omega$. If the set of vertices with finite degree is c.e., then there is a computable partition $G = X_0 \sqcup X_1$ such that neither X_0 nor X_1 is computably isomorphic to G.*

The corollary follows from Theorem 3.3.1 and Proposition 3.1.2.

Corollary 3.3.2. *Let G be a computable graph that has universal vertices but is not isomorphic to K_ω. If the set of vertices with cofinite degree is c.e., then there is a computable partition $G = X_0 \sqcup X_1$ such that neither X_0 nor X_1 is computably isomorphic to G.*

We now give the proof of Theorem 3.3.1.

Proof of Theorem 3.3.1. Without loss of generality, assume the set of vertices of G is ω. If the set of isolated nodes is computable, then let X_0 be the set of isolated nodes and $X_1 = G \setminus X_0$ as in the proof of Theorem 3.1.4. In this case, neither X_0 nor X_1 is even classically isomorphic to G. Therefore, assume the set of isolated nodes is not computable and so, in particular, is infinite.

The construction of the computable partition proceeds in stages with G_s denoting the subgraph of G on vertices $\{0, \ldots, s\}$. At stage s, we determine whether to put s in X_0 or X_1. Following the usual use conventions, if $\Phi_{e,s}(x) = y$, then $x, y < s$, so $x \in G_s$ and y has already been placed in either X_0 or X_1.

For each $e \in \omega$, we need the partition to satisfy the following requirement.

$$R_e : \Phi_e \text{ is not an isomorphism from } G \text{ to } X_0 \text{ or } X_1.$$

We first describe informally our strategy to meet the requirements R_e. Then we give the formal construction.

The strategy for R_e keeps five parameters: numbers i_e and x_e, finite sets D_e and S_e, and a binary string σ_e. The parameter i_e is defined when $\Phi_e(0)$ converges and is set such that $\Phi_e(0) \in X_{i_e}$, indicating we must work to prevent Φ_e from being an isomorphism onto X_{i_e}. Unlike the other parameters, i_e does not change values once it is defined. Notice that i_e may stay undefined: in this case Φ_e is not total and we meet the requirement R_e.

The goal is to make Φ_e map an isolated node in G to a non isolated node in X_{i_e} or vice versa. The parameter x_e marks the isolated node in G we are currently working with. As the value of x_e grows, we attempt to compute the set of isolated nodes in G, defining (and later extending) σ_e to be an initial segment of this computable function. Eventually, because the set of isolated vertices is not computable, σ_e must be wrong about some vertex, and this incorrect vertex will be a diagonalizing value for Φ_e. The sets D_e and S_e contain nodes related to commitments R_e makes about putting future vertices into X_{i_e} or X_{1-i_e} as we try to make specific nodes in X_{i_e} isolated or not.

The construction for a single R_e works as follows. Suppose at stage s_0 we define i_e such that $\Phi_e(0) \in X_{i_e}$. We set x_e to be the least vertex that currently looks isolated in G. Since G_{s_0} might not contain any isolated vertices, we may need to look at vertices in G_t for $t > s_0$ to find a vertex that is isolated in G_t. At this point, we know the vertices $v < x_e$ are not isolated in G, but we are unsure whether x_e is really isolated or not. We record this information by defining σ_e with $|\sigma_e| = x_e$ and $\sigma_e(v) = 0$ for $v < x_e$.

We do nothing more until $\Phi_e(x_e)$ converges. If it does not converge, we meet R_e because Φ_e is not total. Assume $\Phi_e(x_e) = y \in X_{i_e}$ else we win R_e trivially because Φ_e would not be a map onto X_{i_e}. Our goal is to use y to guess whether x_e will be isolated in G in such a way that if our guess is wrong, R_e will be met. Since we are defining X_0 and X_1, we have some control over whether y will be isolated in X_{i_e}.

We split into three substrategies. First, if y already has a neighbor in X_{i_e}, then we know y is not isolated in X_{i_e}. We declare that x_e will not be isolated in G by setting $\sigma_e(x_e) = 0$. If σ_e turns out to be wrong about x_e, then we win R_e because $\Phi_e(x_e) = y$ with x_e isolated in G and y not isolated in X_{i_e}.

Assume y does not currently have a neighbor in X_{i_e}. To determine whether to follow the second or third substrategy, we use the hypothesis that the nodes of finite degree form a c.e. set. In parallel, we enumerate the vertices of finite degree searching for y, and we look ahead in G to see if y gains a future neighbor which is not yet promised to be put in X_0 or X_1. At least one of these searches must succeed as R_e (and later, even higher priority requirements) will only have made finitely many future commitments. The search that terminates first determines which substrategy we follow.

If we see y enumerated in the set of vertices with finite degree, we promise to put all of y's future neighbors into X_{1-i_e} to make y isolated in X_{i_e}. To keep track of this commitment, we place y in D_e. We declare x_e will be isolated by setting $\sigma_e(x_e) = 1$. As long as we keep our promise, if σ_e turns out to be wrong about x_e, then we win R_e because Φ_e maps a non isolated node x_e in G to an isolated node y in X_{i_e}.

If we find a future uncommitted neighbor v of y, we promise to put v into X_{i_e} at stage v and we mark this commitment by putting v into S_e. We declare x_e will not be isolated by setting $\sigma_e(x_e) = 0$. As long as we keep our promise to put v in X_{i_e}, y will not be isolated in X_{i_e}. Therefore, again, we win R_e if σ_e is incorrect about x_e.

Once we have defined $\sigma_e(x_e)$, we repeat the process above. We define $x_{e,s}$ to be the next largest number that currently looks isolated, set $\sigma_e(v) = 0$ for $x_{e,s-1} < v < x_{e,s}$, wait for $\Phi_e(x_e)$ to converge, and employ the appropriate substrategy to define σ_e on the new value of x_e. This process cannot repeat infinitely because the set of isolated nodes is not computable. Therefore, σ_e must be wrong at some stage and we must eventually see a true diagonalization that satisfies R_e.

The strategies for different requirements interact in a standard finite injury way with the priority determined by the index on R_e. If more than one strategy has an opinion about whether to place the vertex s into X_0 or X_1 at stage s, we follow the higher priority strategy and initialize the lower priority one.

One feature to note is that when x_e is defined at stage s, it is currently isolated in G_s (or possibly in G_t for some $t > s$). By the time $\Phi_e(x_e)$ converges, x_e may not longer be isolated. However, that does not make any difference for our strategies. We use the definitions of X_0 and X_1 to force graph theoretic behavior on the image side with no regard to whether the vertex x_e has remained isolated after the stage at which the parameter is assigned.

We give the formal construction. A vertex v is *claimed by* R_e if $v \in S_e$ or v is connected to a vertex in D_e. When R_e is *initialized*, its parameters i_e, x_e and σ_e are undefined and the sets D_e and S_e are set to $\emptyset$. R_e *looks satisfied at stage* s if any of the following are true.

(S1) $\Phi_{e,s}(0)$ diverges.

(S2) $\exists a \neq b < s\,(\Phi_{e,s}(a){\downarrow} = \Phi_{e,s}(b){\downarrow} \vee (\Phi_{e,s}(a) \in X_0 \wedge \Phi_{e,s}(b) \in X_1))$.

(S3) x_e is defined but $\Phi_{e,s}(x_e)$ diverges.

(S4) $\exists x < s\, \exists y \in D_e\,(\Phi_{e,s}(x) = y \wedge x$ is not isolated in $G_s)$.

(S5) $\exists x < s\, \exists y, z\,(x$ is isolated in $G_s \wedge \Phi_{e,s}(x) = y \wedge E(y,z) \wedge z \in S_e)$.

At stage 0, initialize all requirements and put 0 into X_0. At stage $s > 0$, let each R_e with $e < s$ act in order as described in the R_e module below. When these requirements are done acting, check if there is an $e < s$ such that s is claimed by R_e. If not, put s into X_0. If so, let e be the least such index. If $s \in S_e$, put s into X_{i_e} and otherwise put s into X_{1-i_e}. Initialize all R_i with $i > e$ and end the stage. For the R_e module, act in the first case below that applies.

Case 1. R_e *looks satisfied at* s. Do nothing and go to the next requirement.

Case 2. i_e *is not defined.* Since R_e does not look satisfied, $\Phi_{e,s}(0)$ must converge. By use conventions, $\Phi_{e,s}(0) < s$, so $\Phi_e(0)$ is already in X_0 or X_1. Set i_e such that $\Phi_e(0) \in X_{i_e}$ and go to the next requirement.

Case 3. x_e *is not defined.* In this case, σ_e is also undefined. Set $x_{e,s}$ to be the least vertex that is isolated in some G_t for $t \geqslant s$. Define σ_e with $|\sigma_e| = x_e$ and $\sigma_e(v) = 0$ for all $v < x_e$. Go to the next requirement.

Case 4. x_e *is defined.* Since x_e is defined and R_e does not look satisfied, $\Phi_{e,s}(x_e)$ must converge. Let $y_e = \Phi_e(x_e)$ and note that $y_e \in X_{i_e}$.

(4.1) If y_e has a neighbor in X_{i_e}, define $x_{e,s}$ and $\sigma_{e,s}$ as described after (4.2).

(4.2) Otherwise, dovetail the enumerations of the finite degree vertices and of the neighbors of y_e until one of (4.2.1) or (4.2.2) occurs.

(4.2.1) y_e is enumerated as a vertex with finite degree.

(4.2.2) y_e gets a new neighbor that is not protected by a requirement R_i with $i < e$.

If (4.2.1) halts first, put y_e into D_e. If (4.2.2) halts first, put the neighbor into S_e. In either case, define $x_{e,s}$ and $\sigma_{e,s}$ as described below.

Set $x_{e,s}$ to be the least vertex $v > x_{e,s-1}$ that is isolated in G_t for some $t \geqslant s$. Define $\sigma_{e,s}$ to be an extension of $\sigma_{e,s-1}$ of length $x_{e,s}$. If we acted in (4.2.1), set $\sigma_{e,s}(x_{e,s-1}) = 1$, and if we acted in (4.1) or (4.2.2), set $\sigma_{e,s}(x_{e,s-1}) = 0$. In either case, set $\sigma_{e,s}(v) = 0$ for $x_{e,s-1} < v < x_{e,s}$.

This completes the construction. We note some properties that are clear by inspection and that we use implicitly below. First, for any index e and stage s, x_e is defined if and only if σ_e is defined. Second, if $y \in D_e$, then y has finite degree in G. Moreover, if y is placed in D_e at stage s, then y is isolated in G_s. Third, at any stage s, each requirement R_i has claimed only finitely many vertices. Therefore, in (4.2), if y_e has infinite degree, R_e will eventually see a vertex connected to y_e that is not claimed by any R_i with $i < e$.

Lemma 3.3.3. *Consider an index e and a stage s such that $\sigma_{e,s}$ is defined. Let $t < s$ be the last stage at which R_e was initialized. For a vertex $v < |\sigma_{e,s}|$, let $t_v > t$ be the least stage such that $|\sigma_{e,t_v}| > v$. If $v \neq x_{e,t_v-1}$, then v is not isolated in G and $\sigma_{e,t_v}(v) = 0$.*

Proof. By the hypotheses, at stage $t_v - 1$, either σ_e is undefined or $|\sigma_{e,t_v-1}| \leqslant v$. In the former case, R_e acts in Case 3 at t_v to define σ_{e,t_v}, and in the latter case, R_e acts in Case 4 to extend σ_{e,t_v-1} to σ_{e,t_v}. The arguments in each case are essentially the same, so we assume that R_e acts in Case 4.

The parameter x_{e,t_v} is defined to be the least vertex greater than x_{e,t_v-1} that is isolated in G_t for some $t \geqslant t_v$. Fix the stage $t \geqslant t_v$ such that x_{e,t_v} is isolated in G_t. Since $v < x_{e,t_v}$ was not chosen as the value of the parameter, it follows that v is not isolated in G_t and hence is not isolated in G. Furthermore, since $x_{e,t_v-1} < v < x_{e,t_v}$, we set $\sigma_{e,t_v}(v) = 0$. □

Lemma 3.3.4. *For each e, the following properties hold.*

(1) R_e is initialized only finitely often.

(2) There is a stage t such that for all $s \geqslant t$, R_e looks satisfied at s.

Proof. We prove the properties simultaneously by induction on e. First, consider (1). This property holds trivially for R_0. For $e > 0$, fix a stage t such that for all $j < e$, R_j is never initialized after t and R_j looks satisfied at s for all $s \geqslant t$. By construction, the parameters for R_j do not change after t. Since each $y \in D_j$ has finite degree, the sets D_j and S_j can cause a vertex s to be put into X_{i_j} or X_{1-i_j} only finitely often after stage t. In particular, each R_j can only initialize R_e finitely often and so (1) holds for e.

For (2), fix e. Let t be the last stage at which R_e is initialized. Suppose for a contradiction, there is no stage t as in (2). By (S1)-(S3), we must have that $\Phi_e(0)$ converges, Φ_e is 1-1 and maps into X_{i_e}, and $\Phi_e(x_e)$ converges for each value of the parameter x_e after stage t. It follows that, after stage t, x_e takes on an infinite sequence of values $z_0 < z_1 < \ldots$.

Let s_k be the stage at which R_e acts in Case 4 to set $x_{e,s_k} = z_k$ and define σ_{e,s_k} with length x_{e,s_k}. By construction, the sequences σ_{e,s_k} are nested and uniformly computable, so $g_e = \bigcup_k \sigma_{e,s_k}$ is a computable function. To finish the proof, it suffices to show that g_e is the characteristic function for the set of isolated nodes as this provides the desired contradiction.

By Lemma 3.3.3, if $v \neq z_k$ for all k, then v is not isolated in G and $g_e(v) = \sigma_{e,s_\ell}(v) = 0$, where ℓ is least such that $v < z_\ell$. Therefore, g_e is correct on all nodes not of the form z_k.

The value of $g_e(z_k)$ is set at stage s_{k+1} when R_e sees $\Phi_{e,s_{k+1}}(z_k)$ converge and defines $\sigma_{e,s_{k+1}}(z_k)$ in Case 4. Let $y_k = \Phi_e(z_k)$ and note that by (S2), $y_e \in X_{i_e}$. At stage s_{k+1}, R_e either acts in (4.1), puts a neighbor v_k of y_k into S_e via (4.2.2), or puts y_k into D_e via (4.2.1).

Suppose R_e acts in (4.1) or puts v_k in S_e in (4.2.2). By construction, $g_e(z_k) = \sigma_{e,s_{k+1}}(z_k) = 0$, so we need to show z_k is not isolated in G. We claim that y_k will not be isolated in X_{i_e}. If R_e acts in (4.1), then y_k is already not isolated in $X_{i_e,s_{k+1}}$. Otherwise, R_e acts in (4.2.2), and because R_e is not initialized after stage t, the vertex v_k remains in S_e until it is placed in X_{i_e} at stage v_k making y_k not isolated in X_{i_e}. It now follows by (S5) that z_k must eventually get a neighbor in G.

Finally, suppose R_e puts y_k into D_e via (4.2.1). By construction, $g_e(z_k) = \sigma_{e,s_{k+1}}(z_k) = 1$, so we need to show z_k is isolated in G. Since R_e is not initialized again, y_k remains in D_e at all future stages. If z_k were to get a neighbor at a future stage s', then (S4) would be true at every $s \geqslant s'$, contradicting the assumption that there is no stage t as in (2). □

To complete the proof of Theorem 3.3.1, we show each R_e is satisfied. Fix e, let t_0 be the last stage at which R_e is initialized and let $t_1 \geqslant t_0$ be the stage from (2) of Lemma 3.3.4. If R_e looks satisfied for all $s \geqslant t_1$ because of (S1), (S2) or (S3), then R_e is satisfied because Φ_e is either not total, not 1-1, or does not map into a single X_i.

Suppose R_e looks satisfied for all $s \geqslant t_1$ because of (S4) with $\Phi_{e,t_1}(x) = y \in D_e$. The vertex y must have been placed in D_e after stage t_0, so R_e is not initialized after y enters D_e. It follows that no higher priority requirement overrides the R_e commitment to put y's neighbors into X_{1-i_e}. Therefore, eventually all of y's (finitely many) neighbors are in X_{1-i_e}, so y is, in fact, isolated in X_{i_e}. R_e is won because x is not isolated in G but y is isolated in X_{i_e}.

Finally, suppose R_e looks satisfied because of (S5) with $\Phi_{e,t_1}(x) = y$ and an edge $E(y,z)$ with $z \in S_e$. As in the previous paragraph, R_e keeps its commitment to put z into X_{i_e}. Therefore, R_e is met because x is isolated in G and y is not isolated in X_{i_e}. □

3.4 INDUCTION ASPECTS

In the classical proof of Theorem 3.1.4, we used the least number principle $L\Sigma^0_2$ to conclude that if G is not isomorphic to $\mathcal{R}$, then there is a least n such that there exist finite sets A and B with $|A|+|B|=n$ for which the extension axiom $\varphi_{\mathcal{R}}$ fails. In this section, we show that the statement asserting the existence of a least such n is equivalent to $L\Sigma^0_2$.

Let G be a graph. We say that $\langle X_0, X_1\rangle$ is an n-*extension pair* if X_0 and X_1 are disjoint subsets of G with $|X_0|+|X_1|=n$. The *extension property* holds for $\langle X_0, X_1\rangle$ in G if there is a vertex $v \in G$ that is connected to every vertex in X_0 and to none of the vertices in X_1. By definition, G is a random graph if, for all n, every n-extension pair in G has the extension property.

Keep in mind two properties of extension pairs during the construction. First, the only 0-extension pair is $\langle\emptyset,\emptyset\rangle$, which is always satisfied if G is nonempty. Therefore, if $G \not\cong \mathcal{R}$, then the least n for which the extension property fails satisfies $n \geqslant 1$. Second, if G is infinite and has an m-extension pair $\langle X_0, X_1\rangle$ without the extension property, then for every $n \geqslant m$, we can form an n-extension pair without the extension property by adding $n-m$ many vertices from $G \setminus (X_0 \cup X_1)$ to X_0.

Theorem 3.4.1. *The following are equivalent over* RCA_0.

(1) $L\Sigma^0_2$.

(2) For every graph G *not isomorphic to* $\mathcal{R}$*, there is a least* n *for which there is an* n*-extension pair* $\langle X_0, X_1\rangle$ *that fails to have the extension property.*

Proof. (1) implies (2) because the formula with free variable n saying "there is an n-extension pair that fails to have the extension property" is Σ^0_2.

For (2) implies (1), fix a Σ^0_2 formula $\psi(n)$ of the form $\exists x \forall y \phi(n,x,y)$ such that $\psi(n)$ holds for some n. Without loss of generality, we can assume $\neg\psi(0)$ by replacing $\psi(n)$ by $n>0 \wedge \psi(n-1)$ if necessary. In addition, it suffices to construct G such that $(\exists m \leqslant n)\psi(m)$ holds if and only if for some $m \leqslant n$ there is an m-extension pair without the extension property.

We construct G in stages with G_s denoting the graph at stage s. We start by describing informally the construction.

For each $n \geqslant 1$, our strategy tries to ensure that $\psi(n)$ holds if and only if the extension property fails for a pair $\langle F, \emptyset\rangle$ with $|F|=n$. The strategy for n keeps two parameters: x_n and F_n. The parameter x_n is the existential witness we are checking in the formula $\psi(n)$ and F_n is a set of size n. As long as $(\forall y \leqslant s)\phi(n,x_n,y)$ holds, we prevent any node in G_s from connecting to every vertex in F_n. However, if $(\exists y \leqslant s)\neg\phi(n,x_n,y)$, then we add new nodes to witness the extension property for every n-extension pair in G_s including $\langle F_n, \emptyset\rangle$. We increment x_n and add n new elements to form a new set F_n.

This strategy succeeds in isolation. If $\psi(n)$ holds, then x_n eventually reaches a value for which $\forall y \phi(n, x_n, y)$. We choose a final set F_n and prohibit any node from connecting to all of F_n, making the extension property fail for $\langle F_n, \emptyset\rangle$. On the other hand, if $\neg\psi(n)$ holds, then for every value of x_n, there is a stage s such that $(\exists y \leqslant s)\neg\phi(n, x_n, y)$, at which point we increase x_n and add witnesses for all n-extension pairs in G_s. Since each n-extension pair $\langle X_0, X_1\rangle$ in G is contained in G_s for large enough s, the extension property holds for every n-extension pair.

Unfortunately, these strategies interfere with each other. Consider $n_0 < n_1$. When n_1 adds witnesses to realize the extension property for every n_1-extension pair, it adds nodes connected to every point in F_{n_0}. To protect n_0, we restrict n_1 from adding a witness for $\langle X_0, X_1\rangle$ when $F_{n_0} \subseteq X_0$. When $\neg\psi(n_0)$ holds, this restriction has no effect in the limit because the parameter F_{n_0} never settles on a final set. However, when $\psi(n_0)$ holds, it prevents n_1 from realizing the extension property for sets extending the final value of $\langle F_{n_0}, \emptyset\rangle$. It also guarantees there will be n_1-extension pairs without the extension property, a condition that is necessary considering the comments before Theorem 3.4.1.

We give the full construction of G. For each $n \geqslant 1$, we keep parameters $x_{n,s} \in \mathbb{N}$ and $F_{n,s} \subset \mathbb{N}$ with $|F_{n,s}| = n$ defined by primitive recursion on s. We set default values $x_{n,s} = 0$ and $F_{n,s} = \{0, \ldots, n-1\}$ for $n > s$.

We define G_s and m_s by primitive recursion on s. $G_s = \langle V_s, E_s\rangle$ with vertex set $V_s = \{0, \ldots, m_s\}$ and edge relation $E_s \subseteq V_s \times V_s$. The values of m_s are strictly increasing and unbounded, so the vertex set of G is $\bigcup_s V_s = \mathbb{N}$. We maintain $E_{s+1} \upharpoonright V_s = E_s$ so $E_G = \bigcup_s E_s$ is definable with bounded quantifiers:

$$E_G(n, m) \text{ holds if and only if } E_{\max\{n,m\}}(n, m) \text{ holds.}$$

For $s = 0$, set $m_0 = 0$, $V_0 = \{0\}$ and $E_0 = \emptyset$ with the default parameters $x_{1,0} = 0$ and $F_{1,0} = \{0\}$. (This is a throwaway stage because the strategies are only for $n \geqslant 1$.)

For $s = 1$, we set the initial parameters for the $n = 1$ strategy. Set $x_{1,1} = 0$, $m_1 = 1$ (so $V_1 = \{0, 1\}$), $E_1 = \emptyset$ and $F_{1,1} = \{1\}$.

For $s > 1$, we determine which $n < s$ need to act. Define

$$Y_s = \{n : 1 \leqslant n < s \text{ and } (\exists y \leqslant s)\neg\phi(n, x_{n,s-1}, y)\}.$$

If $Y_s = \emptyset$, then we do not act for any $1 \leqslant n < s$. Set $m_s = m_{s-1} + s$ to add s new elements to G_s, but keep $E_s = E_{s-1}$ so there are no edges between the new vertices and the nodes in G_s. Set $x_{s,s} = 0$ and let $F_{s,s} = \{m_{s-1}+1, \ldots, m_s\}$ be the set of s many new elements. For $1 \leqslant n < s$, leave the parameters unchanged: $x_{n,s} = x_{n,s-1}$ and $F_{n,s} = F_{n,s-1}$.

If $Y_s \neq \emptyset$, then we act for each $n \in Y_s$. For $n \in Y_s$, let k_n be the number of n-extension pairs $\langle X_0, X_1 \rangle$ in G_{s-1} such that there is no $m < n$ with $m \notin Y_s$ and $F_{m,s-1} \subseteq X_0$. We refer to such a pair as an *active* n-extension pair.

Set $m_s = m_{s-1} + s + \sum_{n \in Y_s}(k_n + n)$. Use the first $\sum_{n \in Y_s} k_n$ new elements as follows. Order the active n-extension pairs $\langle X_0, X_1 \rangle$ for $n \in Y_s$ and, considering these pairs in order, put edges from the next new element in G_s to the nodes in X_0 (and to no other nodes). These are the only new edges added to G_s.

If $n \notin Y_s$, keep $x_{n,s} = x_{n,s-1}$ and $F_{n,s} = F_{n,s-1}$. If $n \in Y_s$, set $x_{n,s}$ to be the least $x \leqslant s$ such that $(\forall y \leqslant s)\phi(n,x,y)$ holds. If there is no such $x \leqslant s$, then set $x_{n,s} = s+1$. Use the next $\sum_{n \in Y_s} n$ elements to define $F_{n,s}$ for $n \in Y_s$. Consider these n in order, setting $F_{n,s}$ to be the set of the next n many new elements in G_s. Finally, set $x_{s,s} = 0$ and $F_{s,s}$ to be the set of the remaining s many new elements in G_s. This completes the construction at stage s.

By Σ^0_0 induction on s, it follows that for all s and $1 \leqslant n \leqslant s$, $F_{n,s}$ is a set of size n with $x_{n,s} < \min F_{n,s}$ and there is no $z \in G_s$ connected to all the nodes in $F_{n,s}$. In addition, $x_{n,s} \leqslant x_{n,s+1}$, and for any fixed x such that $\forall y \phi(n,x,y)$ holds, we have $x_{n,s} \leqslant x$.

Suppose $\psi(n)$ holds. By $\mathsf{L}\Pi^0_1$ (which holds in RCA_0), there is a least x such that $\forall y \phi(n,x,y)$. Since x is chosen least, $(\forall u < x)\exists y \neg\phi(n,u,y)$ holds, and by $\mathsf{B}\Sigma^0_0$, there is a t such that $(\forall u < x)(\exists y < t)\neg\phi(n,u,y)$. It follows that $x_{n,t} = x$ and $x_{n,s} = x_{n,t} = x$ for every $s \geqslant t$. Therefore, $\lim_s x_{n,s} = x$. Moreover, setting $F_n = F_{n,t}$, we have $F_{n,s} = F_n$ for all $s \geqslant t$ and so $\lim_s F_{n,s} = F_n$.

Similarly, if $\neg\psi(n)$ holds, then the values of $x_{n,s}$ are unbounded as s goes to infinity, as are the values of the minimum elements of $F_{n,s}$.

To complete the proof, we show that for all n, $(\exists m \leqslant n)\psi(m)$ holds if and only if there is an $m \leqslant n$ and an m-extension pair $\langle X_0, X_1 \rangle$ that fails to have the extension property in G.

For the forward direction, fix $n \geqslant 1$ such that $(\exists m \leqslant n)\psi(m)$ holds and fix $m \leqslant n$ with $\psi(m)$. By $\mathsf{L}\Pi^0_1$, fix the least t such that $F_{m,s} = F_{m,t}$ for all $s \geqslant t$ and let $F_m = F_{m,t}$. We show the extension property fails in G for $\langle F_m, \emptyset \rangle$. It suffices to show by Σ^0_0 induction that for all $s \geqslant t$, there is no node in G_s connected to all the nodes in F_m.

For $s = t$, since t is chosen least, the set $F_{m,t} \neq F_{m,t-1}$ consists of m many new elements added to G_t none of which are connected to nodes in G_t. Therefore, the condition holds for $s = t$.

Assume the condition holds for a fixed $s \geqslant t$. A new node $v \in G_{s+1}$ is only connected to a node in G_s when v is used to witness the extension property for a k-extension pair

$\langle X_0, X_1 \rangle$ with $k \in Y_{s+1}$. In this case, ν is only connected to the nodes in X_0. If $k < m$, then $|X_0| \leqslant k$, so ν cannot be connected to every node in F_m. If $k > m$, then by construction, we have $F_{m,s} \not\subseteq X_0$ and hence ν is not connected to every node in F_m.

For the backward direction, assume $(\forall m \leqslant n)\neg\psi(m)$. Fix $m \leqslant n$ and an m-extension pair $\langle X_0, X_1 \rangle$ in G. We need to show this pair has the extension property. Fix t_0 such that $X_0, X_1 \subseteq G_{t_0}$. Suppose $X_0 = \emptyset$. In this case, we need to show there is a node $\nu \in G$ which is not connected to any node in X_1. However, at each stage $s > 0$, we add new elements to G_s that are not connected to any node in G_{s-1}.

Suppose $X_0 \neq \emptyset$ and let ℓ be the least element of X_0. Since $(\forall k < m)\neg\psi(k)$, we have $(\forall k < m)(\forall x \leqslant \ell)\exists y \neg\phi(k, x, y)$. By $B\Sigma^0_0$, we can fix $t > t_0$ such that $(\forall k < m)(\forall x \leqslant \ell)(\exists y < t)\neg\phi(k, x, y)$. It follows that $x_{k,t} > \ell$ for all $k < m$, and hence $\min F_{k,t} > \ell$ for all $k < m$. In particular, for all $s \geqslant t$ and $k < m$, $F_{k,s} \not\subseteq X_0$. Therefore, for any $s > t$ at which we act for m, we add an extension witness for the pair $\langle X_0, X_1 \rangle$ as required. $\square$

4 THE BARRIER RAMSEY THEOREM

This chapter is joint work with Alberto Marcone and Antonio Montalbán.

In this section we use uppercase letters X, Y, Z to denote subsets of $\mathbb{N}$ which may be finite or infinite and we use lowercase letters s, t, u to denote finite subsets of $\mathbb{N}$. We identify a subset of $\mathbb{N}$ with the strictly increasing sequence enumerating it.

It is known since Gödel's first incompleteness theorem that there are true statements about the standard natural numbers that cannot be proved with the axioms of Peano arithmetic. In [PH77] Paris and Harrington proved that a certain statement in finite Ramsey theory, expressible in Peano arithmetic, is not provable in this system. This has been claimed to be the first "natural" statement independent from Peano arithmetic.

Statement 4.0.1 (Paris-Harrington)**.** *Let $n, k, m \in \mathbb{N}$ be such that $m \geqslant n$. There exists $N \in \mathbb{N}$ such that for each $s \subseteq \mathbb{N}$ of cardinality N and each coloring of the n-size subsets of s in k colors, there exists a set $t \subseteq s$ which is homogeneous for the coloring and such that $|t| > \max(m, \min t)$.*

After this seminal result, Ketonen and Solovay [KS81] introduced the notion of α-largeness for $\alpha < \epsilon_0$ (recall that ϵ_0 denotes the first fixed point of the ordinal exponential function in base ω and is the proof theoretic ordinal of Peano arithmetic) and improved the computation of N in Statement 4.0.1. They originally used these largeness notions to give an alternative proof of the unprovability of Statement 4.0.1 from Peano Arithmetic after Paris and Harrington's original proof.

Given a countable ordinal Γ for which we have a system of fundamental sequences (see Definition 1.3.1), we can define what it means for a finite set of natural numbers to be α-large for $\alpha < \Gamma$. Here we mention that typically (though not always, since largeness depends on the chosen system of fundamental sequences) a set s is n-large (for $n \in \omega$) when $|s| \geqslant n$, ω-large if $|s| > \min s$, ω^2-large if it can be partitioned into $\min s$ sets, each entirely preceding the next one and ω-large. Notice that $\alpha < \beta$ does not necessarily imply that every β-large set is also α-large (e.g. $\{2, 5, 8\}$ is ω-large but not n-large for $3 < n < \omega$). In this parlance, Paris-Harrington statement asserts the existence of homogeneous ω-large sets for k-colorings of n-large subsets of a N-large set.

More recently, various authors considered statements extending Statement 4.0.1 to largeness notions and established some relationships between the various parameters [BK99;

BK02; BK06; KPW07; KZ09; KY20; DW10]. All of these results consider α-largeness notions for $\alpha < \epsilon_0$ (except [DW10], which considers $\alpha < \epsilon_\omega$) and Ramsey-like statements that restrict the sizes of the tuples being colored to be n for some fixed natural number n. In this chapter we aim to extend the previous results by studying colorings of all γ-size subsets of some finite set s also when $\gamma \geqslant \omega$. In [PR82; FN08] the infinite Ramsey theorem has been extended to colorings of the γ-size subsets of an infinite set (the homogeneous set is required to be infinite), while [Clo84] gave a computability-theoretic analysis of the effective strength of these results.

We introduce a new, more flexible, framework: largeness notions induced by blocks and barriers (see Definition 1.4.1). Largeness notions induced by blocks generalize largeness notions induced by systems of fundamental sequences. Moreover, each block has a countable ordinal associated to it, its height, which measures the complexity of the block and of its largeness notion.

Notice also that a system of fundamental sequences is defined on some (typically constructive) ordinal Γ and once we fix the system we can work only with ordinals below Γ. On the other hand, we can consider blocks with arbitrary (countable) height.

A useful tool to state results in Ramsey theory is the arrow notation: in Section 4.3 we adapt this notation to our framework and write $\mathcal{B} \to (\mathcal{A})^{\mathcal{C}}_k$ where $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are blocks and $k \in \mathbb{N}$. Then, given ordinals α and γ, we define $\mathrm{Ram}(\alpha)^{\gamma}_k$ as the least β such that for each $\mathcal{A}$ and $\mathcal{C}$ of height respectively α and γ there exists $\mathcal{B}$ of height β with $\mathcal{B} \to (\mathcal{A})^{\mathcal{C}}_k$.

The main result of Chapter 4 is that, for $\gamma < \alpha$ with α infinite, the ordinal $\mathrm{Ram}(\alpha)^{1+\gamma}_{<\omega} = \sup_{k \in \mathbb{N}} \mathrm{Ram}(\alpha)^{1+\gamma}_k$ is

$$\varphi_{\log \gamma}(\alpha \cdot \omega),$$

where $\varphi_{\log \gamma}$ denotes a composition of Veblen functions indexed by the ordinals resulting from a logarithm of γ obtained from its Cantor normal form.

We now describe the structure of Chapter 4. In Section 4.1 we introduce largeness notions induced by systems of fundamental sequences. We also introduce a way to pass from a system of fundamental sequences on some ordinal ζ to a system on Γ_ζ (which is the ζ-th ordinal closed under the binary Veblen function). We also highlight some properties that the new system satisfies. Then we define largeness notions induced by blocks and barriers.

In Section 4.2 we show how to produce blocks and barriers starting from a system of fundamental sequences. Then we show how to obtain what we call a pseudosystem of fundamental sequences starting from a block or a barrier. In both cases, key properties of the involved largeness notions are preserved. This chapter motivates our choice of working with largeness notions induced by barriers instead of systems of fundamental sequences.

In Section 4.3 we introduce the notation for Ramsey theory in the framework with blocks and barriers. We also define the ordinal $\varphi_{\log\gamma}(\alpha \cdot \omega)$ that in the following sections we prove to be $\mathrm{Ram}(\alpha)^{1+\gamma}_{<\omega}$.

Section 4.4 deals with the barrier pigeonhole principle, which computes $\mathrm{Ram}(\alpha)^1_k$. In Section 4.5 we show that under our hypothesis $\varphi_{\log\gamma}(\alpha \cdot \omega) \leqslant \mathrm{Ram}(\alpha)^{1+\gamma}_{<\omega}$: the proof employs the system of fundamental sequences described in Section 4.1. Section 4.6 completes the proof by showing $\mathrm{Ram}(\alpha)^{1+\gamma}_{<\omega} \leqslant \varphi_{\log\gamma}(\alpha \cdot \omega)$: this is obtained for arbitrary countable ordinals α and γ.

Finally, in Section 4.7 we show that our system of fundamental sequences enjoys a crucial property used in the proofs of Section 4.5. Since this proof is rather technical we delay it to the last section.

4.1 LARGENESS NOTIONS...

4.1.1 ...with systems of fundamental sequences

For the rest of Section 4.1.1 we fix a system of fundamental sequences on some ordinal Γ. We also assume to work only with ordinals $< \Gamma$.

Definition 4.1.1. Let $s \in [\mathbb{N}]^{<\omega}$ and α be an ordinal. We denote by $\alpha[s]$ (or $\alpha[s_0, \dots, s_{|s|-1}]$) the ordinal $\alpha[s_0] \dots [s_{|s|-1}]$.

We say that s is *α-large* if $\alpha[s] = 0$, *α-small* if $\alpha[s] > 0$ and *α-size* if s is α-large but $s^* = s \setminus \{\max s\}$ is α-small (α-size are often called "exactly α-large" in the literature).

Notice that every α-large set has a α-size initial segment. We denote by $[X]^\alpha$ the set of α-size subsets of X.

We introduce a sum operation for largeness notions. Given ordinals α and β and a set s, we say that s is $(\alpha \uplus \beta)$-large if s can be partitioned in two parts $s_\beta < s_\alpha$ such that $s = s_\beta {}^\frown s_\alpha$, s_β is β-size and s_α is α-large. We may also say that s is $(\alpha \uplus \beta)$-size if in addition s_α is α-size.

We start by proving a couple of lemmas about nested systems of fundamental sequences (see Definition 1.3.3) .

Lemma 4.1.2. *Assume the system of fundamental sequences is nested. Let $n > 1$ and $(n_i)_{i\in\omega}$ be a sequence such that $n \leqslant n_i$ for each i. Then for each ordinal α there exists $k \in \mathbb{N}$ such that $\alpha[n] = \alpha[n_0, \dots, n_k]$.*

Proof. Fix α and notice that the sequence $(\alpha[n_0,\dots,n_m])_{m\in\omega}$ is strictly decreasing until it reaches 0: let $k \in \mathbb{N}$ be largest such that $\alpha[n_0,\dots,n_k] \geqslant \alpha[n]$ (k exists because $\alpha[n] \leqslant \alpha[n_0]$). We claim that $\alpha[n_0,\dots,n_k] = \alpha[n]$.

For the sake of contradiction suppose that $\alpha[n_0,\dots,n_k] > \alpha[n]$. Then $\alpha > \alpha[n_0,\dots,n_k] > \alpha[n]$ and so by nestedness $\alpha[n_0,\dots,n_k,n] \geqslant \alpha[n]$. Since $n \leqslant n_{k+1}$ then $\alpha[n_0,\dots,n_{k+1}] \geqslant \alpha[n]$, contradicting the maximality of k. □

Lemma 4.1.3. *Assume the system is nested. Let $s,t \in [\mathbb{N}]^{<\omega}$ be such that $|s| \leqslant |t|$ and $1 < t(i) \leqslant s(i)$ for each $i < |s|$. Then $\alpha[t] \leqslant \alpha[s]$ for each ordinal α.*

Proof. We define a strictly increasing function f with $\mathrm{dom}(f) \sqsubseteq \{0,\dots,|t|\}$, $\mathrm{ran}(f) \subseteq \{0,\dots,|s|\}$ and $\alpha[t \restriction i] = \alpha[s \restriction f(i)]$ whenever $f(i)$ is defined.

We start by setting $f(0) = 0$ which satisfies $\alpha[t \restriction 0] = \alpha = \alpha[s \restriction f(0)]$. Assume now that we have defined $f(i)$ and that $\alpha[t \restriction i] = \alpha[s \restriction f(i)]$. Since f is strictly increasing we have that $i \leqslant f(i)$ and so $t(i) \leqslant s(i) \leqslant s(f(i)) \leqslant \dots \leqslant \max s$. By Lemma 4.1.2 we have two cases. Either there exists j with $f(i) \leqslant j < |s|$ such that

$$\alpha[t \restriction i][t(i)] = \alpha[t \restriction i][s(f(i)),\dots,s(j)],$$

so that $\alpha[t \restriction i+1] = \alpha[s \restriction f(i)][s(f(i)),\dots,s(j)] = \alpha[s \restriction j+1]$. In this case we set $f(i+1) = j+1$. Otherwise $\alpha[t \restriction i+1] < \alpha[t \restriction i][s(f(i)),\dots,\max s] = \alpha[s]$: in this case we let $f(i+1)$ undefined and we get $\alpha[t] \leqslant \alpha[t \restriction i+1] < \alpha[s]$ hence our thesis. If f is defined on the whole set $\{0,\dots,|t|\}$ then it must be $|t| = |s|$ and $f(|t|) = |s|$. Therefore $\alpha[t] = \alpha[s]$ and the thesis follows. □

These lemmas show that largeness notions produced by nested systems of fundamental sequences behave as expected. The proof of the following corollary is immediate from Lemma 4.1.3.

Corollary 4.1.4. *Assume the system is nested and let $s \subseteq t$ with $\min t > 1$. If s is α-large for some ordinal α, then t is α-large too. If $s \subsetneq t$ and t is α-size, then s is α-small.*

Given ordinals $\beta < \alpha$, it is not always the case that an α-large set is β-large. To analyze this situation, Ketonen and Solovay developed in [KS81] the notion of $\Rightarrow_n$ and proved that this relation has nice properties on ordinals below ϵ_0. This leads to the fact, proven below, that every α-large set with the minimum larger than a bound depending only on β is β-large.

Definition 4.1.5. Given ordinals α and β we write $\alpha \Rightarrow_n \beta$ to mean that $\beta = \alpha[n,\dots,n]$ for some number (possibly 0) of n's.

Lemma 4.1.6. *Assume the system is nested. If* $\beta \leqslant \alpha$ *there exists* $n > 1$ *such that* $\alpha \Rightarrow_n \beta$.

Proof. Let $n_0, \ldots, n_k$ be a sequence such that $\beta = \alpha[n_0, \ldots, n_k]$: such a sequence exists by letting $n_0 = \min\{n > 1 : \beta \leqslant \alpha[n]\}$ and $n_{i+1} = \min\{n > 1 : \beta \leqslant \alpha[n_0, \ldots, n_i, n]\}$ as long as $\beta < \alpha[n_0, \ldots, n_i]$. Since the sequence $\alpha[n_0, n_1, \ldots]$ is strictly decreasing, we must have at some finite stage $\beta = \alpha[n_0, \ldots, n_k]$. Let $n = \max\{n_0, \ldots, n_k\}$. By Lemma 4.1.2, $\alpha[n_0] = \alpha[n, \ldots, n]$ for a number of n's. Iterating, we get $\beta = \alpha[n_0, \ldots, n_k] = \alpha[n, \ldots, n]$, i.e. $\alpha \Rightarrow_n \beta$. □

Definition 4.1.7. Assume the system is nested. To each ordinal α we associate a natural number $|\alpha|$ that we call the norm of α:

$$|\alpha| = \min\{n > 1 : \Gamma \Rightarrow_n \alpha\}.$$

The norm function was introduced in [KS81] as well.

Proposition 4.1.8. *Assume the system is nested. If* $n \geqslant |\alpha|$ *then* $\Gamma \Rightarrow_n \alpha$.

Proof. Immediate from Lemma 4.1.2. □

The next property of the norm function is crucial: we say that the norm is good because it satisfies this property.

Proposition 4.1.9. *Assume the system is nested. If* $\beta < \delta$ *then* $\beta \leqslant \delta[|\beta|]$.

Proof. Let $n = |\beta|$. For some number of n's we have

$$\Gamma[n, \ldots, n] \geqslant \delta > \Gamma[n, \ldots, n][n] \geqslant \Gamma[n, \ldots, n][n, \ldots, n] = \beta.$$

If $\Gamma[n, \ldots, n] = \delta$ then we immediately get $\delta[n] \geqslant \beta$. On the other hand, if $\Gamma[n, \ldots, n] > \delta$, since we have also $\delta > \Gamma[n, \ldots, n][n]$, then nestedness implies $\delta[n] \geqslant \Gamma[n, \ldots, n][n] \geqslant \beta$. □

Corollary 4.1.10. *Let* $\alpha > \beta$ *and let* s *be such that* $\min s \geqslant |\beta|$. *If* s *is* α*-large then* s *is* β*-large.*

Proof. By Proposition 4.1.9 we have that $\alpha[s \upharpoonright i] > \beta[s \upharpoonright i]$ for each $i < |s|$. In particular it must be $\beta[s] = 0$. □

Notice that the norm function depends on the fixed system of fundamental sequences. For this reason, the goodness is really a property of the system itself. The following important lemmas hold for any good norm.

Lemma 4.1.11. *Assume the system is nested. For any* α *and* β *and* $n \geqslant |\beta|$,

$$\alpha \geqslant \beta \iff \alpha \Rightarrow_n \beta.$$

Proof. The backward direction is obvious.

The forward direction is proved by transfinite induction. We have from Proposition 4.1.9 that if $\alpha > \beta$, then $\alpha[n] \geqslant \alpha[|\beta|] \geqslant \beta$. Then, by the induction hypothesis, we get $\alpha[n] \Rightarrow_n \beta$. □

The following lemma was proved by Bigorajska and Kotlarski in [BK06] for their specific system of fundamental sequences on ϵ_0. We show that any nested system satisfies it.

Lemma 4.1.12 (Estimation Lemma)**.** *Assume the system is nested. If* s *is such that* s^* *is* α*-large (equivalently,* s *is* $(1 \uplus \alpha)$*-large) then there is no strictly decreasing function* $h \colon s \to \alpha$ *satisfying* $|h(s_i)| \leqslant s_i$ *for all* $i < |s|$.

Proof. Let $\langle s_0, \dots, s_n \rangle$ be the α-size proper prefix of s. Suppose $h \colon s \to \alpha$ is strictly decreasing and satisfies $|h(s_i)| \leqslant s_i$ for all $i < |s|$.

We prove by induction on $i \leqslant n$ that $h(s_i) \leqslant \alpha[s_0, \dots, s_i]$. When $i = 0$, $h(s_0) < \alpha$ and $|h(s_0)| \leqslant s_0$ imply, by Proposition 4.1.9, $h(s_0) \leqslant \alpha[|h(s_0)|] \leqslant \alpha[s_0]$. For the successor step, we have $h(s_i) \leqslant \alpha[s_0, \dots, s_i]$ by induction hypothesis, and hence $h(s_{i+1}) < \alpha[s_0, \dots, s_i]$ because h is strictly decreasing. Since we are assuming $|h(s_{i+1})| \leqslant s_{i+1}$, using Proposition 4.1.9 we have $h(s_{i+1}) \leqslant \alpha[s_0, \dots, s_i][|h(s_{i+1})|] \leqslant \alpha[s_0, \dots, s_{i+1}]$.

We thus obtain $h(s_n) \leqslant \alpha[s_0, \dots, s_n] = 0$ and so it cannot be $h(s_{n+1}) < h(s_n)$. □

We now define a way of going from a system of fundamental sequences on an ordinal ζ to a system of fundamental sequences on the ordinal Γ_ζ – the ζ-th fixed point for the binary Veblen function.

Definition 4.1.13. Given a system of fundamental sequences on some ordinal ζ (which we denote as $\xi\lceil n\rceil$ for $\xi < \zeta$), we define a system of fundamental sequences on Γ_ζ as follows:

(1) if $\alpha = \alpha_0 + \omega^{\alpha_1}$ with $\alpha_0 \ggg \omega^{\alpha_1}$ then $\alpha[n] = \alpha_0 + (\omega^{\alpha_1}[n])$,

(2) $0[n] = 0$ and $1[n] = 0$,

(3) If $0 < \alpha < \omega^\alpha$ then $\omega^\alpha[n] = \omega^{\alpha[n]} \cdot n$,

(4) If $0 < \delta < \varphi_\delta(0)$ then $\varphi_\delta(0)[n] = \varphi^{n+1}_{\delta[n]}(0)$,

(5) If $0 < \alpha < \varphi_\delta(\alpha)$ and $0 < \delta$ then $\varphi_\delta(\alpha)[n] = \varphi^{n+1}_{\delta[n]}(\varphi_\delta(\alpha[n]) + 1)$,

(6) $\Gamma_0[n] = \varphi_{\varphi_{\cdot_{\cdot_{\cdot \varphi_0(0)}}\cdot^{\cdot^{\cdot}}(0)}}(0)$, where φ is iterated $n+1$ times,

(7) If $0 < \xi < \zeta$ then $\Gamma_\xi[n] = \varphi_{\varphi_{\cdot_{\cdot_{\cdot \varphi_{\Gamma_{\xi\lceil n\rceil}+1}(0)}}\cdot^{\cdot^{\cdot}}(0)}}(0)$, with φ iterated $n+1$ times.

Notice that (7) above is where we are explicitly using the system of fundamental sequences $\xi[n]$ on ζ. It is not hard to see that Definition 4.1.13 really yields a system of fundamental sequences. Moreover, this system is regular (see Definition 1.3.2).

Theorem 4.1.14. *If the system of fundamental sequences on ζ is nested then the system of fundamental sequences on Γ_ζ introduced in Definition 4.1.13 is nested.*

We postpone the quite technical proof of Theorem 4.1.14 to Section 4.7. We tacitly assume from now on that the system of Definition 4.1.13 is nested.

Remark 4.1.15. Notice that our system fails the Bachmann property just by the definition of the fundamental sequences for successors and powers of ω. Let β_0 be a limit ordinal which is not an ϵ-ordinal (namely, not a fixed point of the exponential function in base ω φ_0) and $n > 1$. Let $\gamma_0 = \beta_0[n] + 1$, so that in particular $\gamma_0 < \beta_0$. Let $\beta = \omega^{\beta_0}$ and $\gamma = \omega^{\gamma_0}$: then $\beta[n] = \omega^{\beta_0[n]} \cdot n$ and $\gamma[1] = \omega^{\beta_0[n]}$, so that $\beta > \gamma > \beta[n] > \gamma[1]$.

Moreover, notice that without the requirement $n > 1$ in Definition 1.3.3 our system cannot be nested. In fact, for $\gamma = \omega^{\omega^{\omega^{\epsilon_0+1}}}$, $\beta = \epsilon_1$ and $n = 1$ we have $\gamma[1] = \omega^{\omega^{\omega^{\epsilon_0}}} = \epsilon_0 < \beta[1] = \omega^{\omega^{\epsilon_0+1}} < \gamma < \beta$.

4.1.2 ...with blocks and barriers

We described fronts, blocks and barriers in Definition 1.4.1. We also observed that we can associate to a front $\mathcal{B}$ two ordinals: its order type $\text{o.t.}(\mathcal{B})$ and its height $\text{ht}(\mathcal{B})$. We now establish the relationship between the order type and the height of a smooth barrier.

Proposition 4.1.16. *If $\mathcal{B}$ is a smooth barrier then $\text{o.t.}(\mathcal{B}) = \omega^\alpha$ if and only if $\text{ht}(\mathcal{B}) = \alpha$.*

Proof. If $\mathcal{B}$ is the degenerate front then the equivalence is immediate with $\alpha = 0$.

Now assume $\text{o.t.}(\mathcal{B}) = \omega^\alpha$ with $\alpha > 0$ and suppose the result holds for all smooth barriers of order type less than ω^α. Notice that $\langle n \rangle \in T(\mathcal{B})$ for every n. Since $\mathcal{B}$ is a smooth barrier, each $\mathcal{B}_n$ is a smooth barrier (possibly degenerate) and so $\text{o.t.}(\mathcal{B}_n) = \omega^{\gamma_n}$ for some γ_n. Since $\omega^\alpha = \sum_{n\in\omega} \omega^{\gamma_n}$ we have $\sup\{\gamma_n + 1 : n \in \omega\} = \alpha$. By inductive hypothesis $\text{ht}(\langle n \rangle) = \text{ht}(\mathcal{B}_n) = \gamma_n$ and therefore $\text{ht}(\mathcal{B}) = \sup\{\gamma_n + 1 : n \in \omega\} = \alpha$.

If instead $\text{ht}(\mathcal{B}) = \alpha$ then, since $\mathcal{B}$ is a smooth barrier, $\text{o.t.}(\mathcal{B}) = \omega^\gamma$ for some γ. By the direction already proved, it must be $\gamma = \alpha$. □

We noticed in Section 1.4 that for each countable ordinal β there exists a smooth barrier of order type ω^β. By Proposition 4.1.16, for each countable ordinal β there exists a smooth barrier of height β.

A nice property satisfied by each smooth barrier $\mathcal{B}$ (but not by every barrier) is that the height is preserved by restricting to final segments of $\text{base}(\mathcal{B})$.

Lemma 4.1.17. *If $\mathcal{B}$ is a smooth barrier and X is a final segment of $\text{base}(\mathcal{B})$ then $\text{ht}(\mathcal{B}) = \text{ht}(\mathcal{B} \restriction X)$.*

Proof. It is clear that $\text{ht}(\mathcal{B}) \geqslant \text{ht}(\mathcal{B} \restriction X)$. Let $\alpha = \text{ht}(\mathcal{B})$, so that by Proposition 4.1.16 $\text{o.t.}(\mathcal{B}) = \omega^\alpha$. Since $\mathcal{B} \restriction X$ is a final segment of $\mathcal{B}$ with respect to the lexicographic order we have $\text{o.t.}(\mathcal{B} \restriction X) = \omega^\alpha$ and hence $\text{ht}(\mathcal{B} \restriction X) = \alpha$. □

The first part of the following lemma is [Mar94, Lemma 3.2], while the second follows from the first and Proposition 4.1.16.

Lemma 4.1.18. *If $\mathcal{A}$ and $\mathcal{B}$ are fronts with the same base such that $\text{o.t.}(\mathcal{A}) < \text{o.t.}(\mathcal{B})$ then there exist $s \in \mathcal{A}$ and $t \in \mathcal{B}$ such that $s \sqsubset t$.*

If $\mathcal{A}$ and $\mathcal{B}$ are smooth barriers with the same base such that $\text{ht}(\mathcal{A}) < \text{ht}(\mathcal{B})$ then there exist $s \in \mathcal{A}$ and $t \in \mathcal{B}$ such that $s \sqsubset t$.

The next lemma gives an alternative characterization of smoothness (recall Definition 1.4.1).

Lemma 4.1.19. *Let $\mathcal{B}$ be a block. The following are equivalent:*

(1) $\mathcal{B}$ is smooth,

(2) for all $s, t \in T(\mathcal{B})$ if $|s| = |t|$ and $\text{ht}(s) < \text{ht}(t)$ then there exists $i < |s|$ such that $s(i) < t(i)$.

Proof. First assume (2). Let $s, t \in \mathcal{B}$ be such that $|s| < |t|$. Let $u \sqsubset t$ be its initial segment of length $|s|$. Then $\text{ht}(u) > 0$ and so by (2) there exists $i < |s|$ such that $s(i) < u(i) = t(i)$ meaning that $\mathcal{B}$ is smooth.

Now assume that $\mathcal{B}$ is smooth and fix $s, t \in T(\mathcal{B})$ of the same length and such that $\text{ht}(s) < \text{ht}(t)$. We proceed by induction on $\text{ht}(s)$. If $\text{ht}(s) = 0$ then $\text{ht}(t) > 0$ and so $s \in \mathcal{B}$ and $t \in T(\mathcal{B}) \setminus \mathcal{B}$. Let $u \in \mathcal{B}$ be an extension of t. Then $|s| = |t| < |u|$ and so by (1) there exists $i < |s|$ such that $s(i) < u(i) = t(i)$. Suppose now that $\text{ht}(s) = \alpha > 0$. Let $n > \max(s \cup t)$ be such that $\text{ht}(t^\frown\langle n\rangle) \geqslant \alpha$. Then $\text{ht}(s^\frown\langle n\rangle) < \alpha \leqslant \text{ht}(t^\frown\langle n\rangle)$. By inductive hypothesis there exists $i < |s^\frown\langle n\rangle|$ such that $s^\frown\langle n\rangle(i) < t^\frown\langle n\rangle(i)$. Notice that it must be $i < |s|$ because $s^\frown\langle n\rangle(|s|) = n = t^\frown\langle n\rangle(|s|)$. Therefore $s(i) < t(i)$. □

We now define largeness notions starting from fronts. In Section 4.2 we show that these are reasonable generalizations of the classical largeness notions based on systems of fundamental sequences.

Definition 4.1.20. If $\mathcal{B}$ is a front, we say that $t \in [\text{base}(\mathcal{B})]^{<\omega}$ is $\mathcal{B}$-*large* if there exists $s \in \mathcal{B}$ such that $s \sqsubseteq t$, $\mathcal{B}$-*small* if there exists $s \in \mathcal{B}$ such that $t \sqsubset s$, and $\mathcal{B}$-*size* if $t \in \mathcal{B}$.

If $\mathcal{B}$ is the degenerate front then all finite sets are $\mathcal{B}$-large and the only $\mathcal{B}$-size set is $t = \langle\rangle$. If $\mathcal{B} = [\mathbb{N}]^k$ then the $\mathcal{B}$-large sets are exactly the finite sets of cardinality at least k. If $\mathcal{B}$ is the Schreier barrier then $\mathcal{B}$-large means ω-large in the classical (Paris-Harrington[1]) sense.

Notice that in the context of largeness notions with smooth barriers, the $\oplus$ operation (see Definition 1.4.3) plays essentially the same role as the $\uplus$ operation we defined for largeness notions associated to systems of fundamental sequences in Subsection 4.1.1.

When we consider barriers, we immediately obtain the analogue of Corollary 4.1.4.

Corollary 4.1.21. *If $\mathcal{B}$ is a barrier, $s \subseteq t$ and s is $\mathcal{B}$-large, then t is $\mathcal{B}$-large too. If $s \subsetneq t$ and t is $\mathcal{B}$-size, then s is $\mathcal{B}$-small.*

Given a front $\mathcal{B}$ we can write $\text{ht}(\mathcal{B})$ in Cantor normal form as $\omega^{\beta_0} + \ldots + \omega^{\beta_{n-1}}$. It is however not always the case, even assuming smoothness of $\mathcal{B}$, that we can decompose $\mathcal{B}$ as the sum of smooth barriers $\mathcal{B}_i$ of height ω^{β_i} for $i < n$. It is therefore natural to give the following definition.

Definition 4.1.22. Let $\mathcal{B}$ be a smooth barrier of height (written in Cantor normal form) $\omega^{\beta_0} + \ldots + \omega^{\beta_{n-1}}$. We say that $\mathcal{B}$ is *decomposable* if there exist smooth barriers $\mathcal{B}_0, \ldots, \mathcal{B}_{n-1}$ with $\text{ht}(\mathcal{B}_i) = \omega^{\beta_i}$ such that $\mathcal{B} = \mathcal{B}_0 \oplus \ldots \oplus \mathcal{B}_{n-1}$. The ordered sequence of barriers $\mathcal{B}_0, \ldots, \mathcal{B}_{n-1}$ is called a *decomposition* of $\mathcal{B}$. We abbreviate smooth decomposable barrier with SD-barrier.

Notice that a smooth barrier with indecomposable height is already a decomposition of itself.

We now describe how to extend the elements of a smooth barrier to obtain a SD-barrier with the same base and height. This requires working with the terms of the Cantor normal form of the height one by one. Recall that if ω^{β_1} is the last term in the Cantor normal form of $\text{ht}(\mathcal{B})$ (which has more than one term) then we can write $\text{ht}(\mathcal{B}) = \beta_0 + \omega^{\beta_1}$ with $\beta_0 \ggg \omega^{\beta_1}$.

Definition 4.1.23. Let $\mathcal{B}$ be a non degenerate front with $\text{ht}(\mathcal{B}) = \beta_0 + \omega^{\beta_1}$ where $\beta_0 \ggg \omega^{\beta_1}$. The *Lower Part of* $\mathcal{B}$ is

$$\overline{\mathcal{B}}_1 = \{t \in T(\mathcal{B}) : \text{ht}_{\mathcal{B}}(t) \leqslant \beta_0 \wedge \text{ht}_{\mathcal{B}}(t^*) > \beta_0\}.$$

[1] see Statement 4.0.1.

Let

$$T_0 = \bigcup_{t\in\overline{\mathcal{B}}_1} T(\mathcal{B}_t) \cup \{\langle n\rangle : n \in \text{base}(\mathcal{B}) \setminus \bigcup_{t\in\overline{\mathcal{B}}_1} \text{base}(\mathcal{B}_t)\}.$$

The *Upper Part of* $\mathcal{B}$ is the set $\overline{\mathcal{B}}_0$ of leaves of T_0.

We remark that $\{\langle n\rangle : n \in \text{base}(\mathcal{B}) \setminus \bigcup_{t\in\overline{\mathcal{B}}_1} \text{base}(\mathcal{B}_t)\}$ is included in the definition of T_0 just to ensure $\text{base}(\overline{\mathcal{B}}_0) = \text{base}(\mathcal{B})$.

Lemma 4.1.24. *Let $\mathcal{B}$ be a smooth barrier with $\text{ht}(\mathcal{B}) = \beta_0 + \omega^{\beta_1}$ where $\beta_0 \gg \omega^{\beta_1}$. Then $\overline{\mathcal{B}}_0$ and $\overline{\mathcal{B}}_1$ are smooth barriers with the same base as $\mathcal{B}$, $\text{ht}(\overline{\mathcal{B}}_0) = \beta_0$, $\text{ht}(\overline{\mathcal{B}}_1) = \omega^{\beta_1}$ and each element of $\overline{\mathcal{B}}_0 \oplus \overline{\mathcal{B}}_1$ extends some element of $\mathcal{B}$.*

Proof. It is clear that $\overline{\mathcal{B}}_1$ is prefix free. For each $n \in \text{base}(\mathcal{B})$, $\langle n\rangle \in T(\overline{\mathcal{B}}_1)$ since $\langle n\rangle^* = \langle\rangle$ and $\text{ht}_{\mathcal{B}}(\langle\rangle) = \beta_0 + \omega^{\beta_1} > \beta_0$. Thus $\text{base}(\overline{\mathcal{B}}_1) = \text{base}(\mathcal{B})$.

Let $X \subseteq \text{base}(\mathcal{B})$ be infinite and let $s \in \mathcal{B}$ be such that $s \sqsubset X$. Let $u \sqsubseteq s$ be longest such that $\text{ht}_{\mathcal{B}}(u) \leqslant \beta_0$. Notice that $u \neq \langle\rangle$ and hence $u \in \overline{\mathcal{B}}_1$. Thus $\overline{\mathcal{B}}_1$ is a block.

Before proving the smoothness of $\overline{\mathcal{B}}_1$ we show that for each $s \in T(\overline{\mathcal{B}}_1)$

$$\text{ht}_{\overline{\mathcal{B}}_1}(s) = \text{ht}_{\mathcal{B}}(s) \dot{-} \beta_0 = \begin{cases} 0 & \text{if } \text{ht}_{\mathcal{B}}(s) < \beta_0 \\ \text{ht}_{\mathcal{B}}(s) - \beta_0 & \text{if } \text{ht}_{\mathcal{B}}(s) \geqslant \beta_0 \end{cases}$$

Notice that in the second case the Cantor normal form of $\text{ht}_{\mathcal{B}}(s)$ starts with β_0 and so we are simply removing those terms. We proceed by induction on $\text{ht}_{\overline{\mathcal{B}}_1}(s)$. If $\text{ht}_{\overline{\mathcal{B}}_1}(s) = 0$ then $s \in \overline{\mathcal{B}}_1$ and $\text{ht}_{\mathcal{B}}(s) \leqslant \beta_0$ so that $\text{ht}_{\mathcal{B}}(s) \dot{-} \beta_0 = 0$. If $\text{ht}_{\overline{\mathcal{B}}_1}(s) > 0$ then $s \in T(\overline{\mathcal{B}}_1) \setminus \overline{\mathcal{B}}_1$ and

$$\begin{aligned} \text{ht}_{\overline{\mathcal{B}}_1}(s) &= \sup_{n\in\text{base}(\mathcal{B})} (\text{ht}_{\overline{\mathcal{B}}_1}(s^\frown\langle n\rangle) + 1) \\ &= \sup_{n\in\text{base}(\mathcal{B})} (\text{ht}_{\mathcal{B}}(s^\frown\langle n\rangle) \dot{-} \beta_0 + 1) \\ &= \sup_{n\in\text{base}(\mathcal{B})} (\text{ht}_{\mathcal{B}}(s^\frown\langle n\rangle) + 1 \dot{-} \beta_0) \\ &= \sup_{n\in\text{base}(\mathcal{B})} (\text{ht}_{\mathcal{B}}(s^\frown\langle n\rangle) + 1) \dot{-} \beta_0 \\ &= \text{ht}_{\mathcal{B}}(s) \dot{-} \beta_0, \end{aligned}$$

where we are using the above observation about $\text{ht}_{\mathcal{B}}(s) \dot{-} \beta_0$.

In particular we have $\text{ht}(\overline{\mathcal{B}}_1) = \text{ht}_{\overline{\mathcal{B}}_1}(\langle\rangle) = \text{ht}_{\mathcal{B}}(\langle\rangle) \dot{-} \beta_0 = \omega^{\beta_1}$.

To prove the smoothness of $\overline{\mathcal{B}}_1$ we show (2) of Lemma 4.1.19. Let $s, t \in T(\overline{\mathcal{B}}_1)$ be such that $|s| = |t|$ and $\text{ht}_{\overline{\mathcal{B}}_1}(s) < \text{ht}_{\overline{\mathcal{B}}_1}(t)$. We claim that $\text{ht}_{\mathcal{B}}(s) < \text{ht}_{\mathcal{B}}(t)$. Towards a contradiction, suppose that $\text{ht}_{\mathcal{B}}(s) \geqslant \text{ht}_{\mathcal{B}}(t)$. If $\text{ht}_{\mathcal{B}}(t) \geqslant \beta_0$ then both Cantor normal forms start with

β_0. When we perform $\dot{-}\beta_0$ we simply remove those terms and get $\text{ht}_{\overline{\mathcal{B}}_1}(s) \geqslant \text{ht}_{\overline{\mathcal{B}}_1}(t)$, a contradiction. If $\text{ht}_{\mathcal{B}}(s) \geqslant \beta_0 > \text{ht}_{\mathcal{B}}(t)$ or $\beta_0 > \text{ht}_{\mathcal{B}}(s)$ then $\text{ht}_{\overline{\mathcal{B}}_1}(t) = 0$ and so $\text{ht}_{\overline{\mathcal{B}}_1}(s) \geqslant \text{ht}_{\overline{\mathcal{B}}_1}(t)$, a contradiction. Therefore $\text{ht}_{\mathcal{B}}(s) < \text{ht}_{\mathcal{B}}(t)$. By smoothness of $\mathcal{B}$ and Lemma 4.1.19, there exists $i < |s|$ such that $s(i) < t(i)$ and so $\overline{\mathcal{B}}_1$ is smooth.

We already noticed that $\text{base}(\overline{\mathcal{B}}_0) = \text{base}(\mathcal{B})$. We now show that $\overline{\mathcal{B}}_0$ is a block. Recall that for each $t \in T(\mathcal{B})$, $\mathcal{B}_t$ is a smooth barrier of height $\text{ht}_{\mathcal{B}}(t)$ and base either $\mathbb{N}$ or $\text{base}(\mathcal{B}) \setminus \{0, \dots, \max t\}$. Notice that $\bigcup_{t \in \overline{\mathcal{B}}_1} T(\mathcal{B}_t)$ is infinite since there exists $s \in \overline{\mathcal{B}}_1 \setminus \mathcal{B}$ and so for each $n \in \text{base}(\mathcal{B})$ with $n > \max s$, $\langle n \rangle \in T(\mathcal{B}_s)$. Let $X \subseteq \text{base}(\overline{\mathcal{B}}_0)$ be infinite. If $\langle \min X \rangle \notin \bigcup_{t \in \overline{\mathcal{B}}_1} T(\mathcal{B}_t)$ then we are done as $\langle \min X \rangle \in \overline{\mathcal{B}}_0$. Otherwise, for each $t \in \overline{\mathcal{B}}_1$ with $\max t < \min X$, $X \subseteq \text{base}(\mathcal{B}_t)$. For each such t, there exists a prefix $s_t \in \mathcal{B}_t$ of X. Since there are only finitely many such t, the longest s_t belongs to $\overline{\mathcal{B}}_0$. We conclude that $\overline{\mathcal{B}}_0$ is a block with the same base as $\mathcal{B}$.

Next we prove that $\overline{\mathcal{B}}_0$ is smooth. Let $s, t \in \overline{\mathcal{B}}_0$ with $|s| < |t|$. If $s(0) < t(0)$ we are done, so assume that $t(0) \leqslant s(0)$. Since $|t| > 1$ there exists $u \in \overline{\mathcal{B}}_1$ such that $t \in \mathcal{B}_u$. Since $t(0) \leqslant s(0)$ we know that $s \subset \text{base}(\mathcal{B}_u)$. Notice that s must be $\mathcal{B}_u$-large since $s \in \overline{\mathcal{B}}_0$ and $\mathcal{B}_u \subset T(\overline{\mathcal{B}}_0)$. Therefore there exists $s' \sqsubseteq s$ such that $s' \in \mathcal{B}_u$ and by smoothness of $\mathcal{B}_u$ we obtain that there exists $i < |s'|$ such that $s(i) = s'(i) < t(i)$ as required.

Our next goal is to show that $\text{ht}(\overline{\mathcal{B}}_0) = \beta_0$. Let $t \in T(\mathcal{B})$ be such that $\text{ht}_{\mathcal{B}}(t) = \beta_0$ and notice that $t \in \overline{\mathcal{B}}_1$. Therefore $\text{ht}(\overline{\mathcal{B}}_0) \geqslant \text{ht}(\mathcal{B}_t) = \beta_0$. Now let $n \in \text{base}(\overline{\mathcal{B}}_0)$. If $n \notin \bigcup_{t \in \overline{\mathcal{B}}_1} \text{base}(\mathcal{B}_t)$ then $\text{ht}_{\overline{\mathcal{B}}_0}(\langle n \rangle) = 0 < \beta_0$. Otherwise there are finitely many $t \in \overline{\mathcal{B}}_1$ with $\langle n \rangle \in T(\mathcal{B}_t)$. Call $\mathcal{A}$ the smooth barrier consisting of the $\sqcup$-union of $\mathcal{B}_t$ for these t's. Then $T(\mathcal{A}) \subseteq T(\overline{\mathcal{B}}_0)$ and $\text{ht}_{\overline{\mathcal{B}}_0}(\langle n \rangle) = \text{ht}_{\mathcal{A}}(\langle n \rangle) = \max\{\text{ht}_{\mathcal{B}_t}(\langle n \rangle) : t \in \overline{\mathcal{B}}_1 \wedge \langle n \rangle \in T(\mathcal{B}_t)\} < \beta_0$ by Remark 1.4.4 and definition of $\overline{\mathcal{B}}_1$. This implies that $\text{ht}(\overline{\mathcal{B}}_0) \leqslant \beta_0$.

We are left to prove that each element of $\overline{\mathcal{B}}_0 \oplus \overline{\mathcal{B}}_1$ extends some element of $\mathcal{B}$. Let $t^\frown s \in \overline{\mathcal{B}}_0 \oplus \overline{\mathcal{B}}_1$ with $t \in \overline{\mathcal{B}}_1$ and $s \in \overline{\mathcal{B}}_0$. Towards a contradiction, suppose that $t^\frown s \in T(\mathcal{B}) \setminus \mathcal{B}$ and let s' be such that $s \sqsubset s'$ and $t^\frown s' \in \mathcal{B}$. Then $s' \in \mathcal{B}_t$ and so some extension of s' belongs to $\mathcal{B}_0$. Since we are assuming $s \in \overline{\mathcal{B}}_0$, this contradicts that $\mathcal{B}_0$ is a block. □

Notice that in general it is not the case that $\overline{\mathcal{B}}_0 \oplus \overline{\mathcal{B}}_1 = \mathcal{B}$ as there can be $t_1, t_2 \in \overline{\mathcal{B}}_1$ and $s_1 \sqsubset s_2$ such that $s_1 \in \mathcal{B}_{t_1}$ and $s_2 \in \mathcal{B}_{t_2}$.

Example 4.1.25. Let $\mathcal{B}$ be a smooth barrier with base $\mathbb{N}$ of height $\omega + 1$ such that $\langle 0 \rangle \in \mathcal{B}$ and for each $s \in \mathcal{B}$ if $\min s > 0$ then $|s| > 1$. Then by definition the Lower Part $\overline{\mathcal{B}}_1$ has height 1 and must be the smooth barrier $\mathbb{1}$ of singletons on base $\mathbb{N}$. In particular $\langle 0 \rangle \in \overline{\mathcal{B}}_1$. The Upper Part $\overline{\mathcal{B}}_0$ clearly contains sequences with positive minimum. Then $\langle 0 \rangle \notin \overline{\mathcal{B}}_0 \oplus \overline{\mathcal{B}}_1$.

Corollary 4.1.26. *If $\mathcal{B}$ is a smooth barrier, then there exists a* SD*-barrier $\mathcal{B}'$ such that* $\text{base}(\mathcal{B}) = \text{base}(\mathcal{B}')$, $\text{ht}(\mathcal{B}) = \text{ht}(\mathcal{B}')$ *and* $\mathcal{B} \subset T(\mathcal{B}')$.

Proof. We proceed by induction on the number of terms in the Cantor normal form of $\mathrm{ht}(\mathcal{B})$. If there is only one term the result is trivial.

Suppose that the Cantor normal form of $\mathrm{ht}(\mathcal{B})$ has at least two terms. By Lemma 4.1.24, the Cantor normal form of $\mathrm{ht}(\overline{\mathcal{B}}_0)$ has one less term than the one of $\mathrm{ht}(\mathcal{B})$. Therefore, by inductive hypothesis, there exists a SD-barrier $\overline{\mathcal{B}}_0'$ with the same base and same height as $\overline{\mathcal{B}}_0$ and such that $\overline{\mathcal{B}}_0 \subset T(\overline{\mathcal{B}}_0')$. It is clear that $\mathcal{B} \subset T(\overline{\mathcal{B}}_0 \oplus \overline{\mathcal{B}}_1) \subseteq T(\overline{\mathcal{B}}_0' \oplus \overline{\mathcal{B}}_1)$ and that $\overline{\mathcal{B}}_0' \oplus \overline{\mathcal{B}}_1$ is a SD-barrier as required. □

Notice that by Lemma 1.4.2 and by Corollary 4.1.26, starting from a front $\mathcal{B}$ we can always produce a SD-barrier $\mathcal{B}'$ such that $\mathrm{base}(\mathcal{B}) = \mathrm{base}(\mathcal{B}')$, $\mathrm{ht}(\mathcal{B}) = \mathrm{ht}(\mathcal{B}')$ and $\mathcal{B} \subset T(\mathcal{B}')$.

Notice that the only SD-barriers of height n are of the form $[\mathrm{base}(\mathcal{B})]^n$. Hence, up to isomorphism, there is only one such a SD-barrier.

Recall that there exist smooth barriers of arbitrary countable height. Let $\beta > 0$ be a countable ordinal with Cantor normal form $\omega^{\beta_0} + \ldots + \omega^{\beta_n}$. For each $i \leqslant n$ fix a smooth barrier $\mathcal{B}_i$ of height ω^{β_i} and assume that for each $i < n$ $\mathrm{base}(\mathcal{B}_{i+1})$ is a final segment of $\mathrm{base}(\mathcal{B}_i)$. Hence by Remark 1.4.4 $\mathcal{B}_0 \oplus \ldots \oplus \mathcal{B}_n$ is a SD-barrier of height β. Therefore there exist SD-barriers of arbitrary countable height.

Let α and β be ordinals such that α is strictly less than the first term in the Cantor normal form of β. Fix smooth barriers $\mathcal{A}$ of height α and $\mathcal{B}$ of height β. Then $\mathcal{A} \oplus \mathcal{B}$ is a smooth barrier of height $\alpha + \beta = \beta$ which is not isomorphic to $\mathcal{B}$. It follows that for each ordinal $\beta \geqslant \omega$ there are infinitely many non isomorphic smooth barriers of height β. Therefore there are infinitely many non isomorphic SD-barriers of any infinite height.

4.2 BRIDGES BETWEEN SYSTEMS AND BARRIERS

In this section we show that given a system of fundamental sequences, we can build fronts which yield the same largeness notions. Moreover, if the system is nested and regular, we get SD-barriers. Next we show the converse, namely that given a front we can produce something very similar to a system of fundamental sequences. Again, if the front is a SD-barrier, the induced system satisfies weaker versions of nestedness and regularity.

For the rest of this section, fix a system of fundamental sequences on some ordinal Γ and an infinite set $M \subseteq \mathbb{N}$ with $\min M > 1$[2]. If $\alpha < \Gamma$, let $\mathcal{B}^\alpha = [M]^\alpha$ be the set of α-size subsets of M.

[2]the restriction to natural numbers larger than 1 mirrors our definition of nestedness.

Theorem 4.2.1. *For each $\alpha < \Gamma$, $\mathcal{B}^\alpha$ is a front. Moreover, if the system is nested every $\mathcal{B}^\alpha$ is a smooth barrier.*

Proof. We first need to check the three properties of Definition 1.4.1. If $\alpha = 0$ then $\mathcal{B}^\alpha$ is easily seen to be the degenerate front, so suppose $\alpha > 0$.

(1) Let $n \in M$ and $n_0, \dots, n_{k-1} \in M$ so that $\alpha[n, n_0, \dots, n_{k-1}] = 0$ but $\alpha[n, n_0, \dots, n_{k-2}] > 0$. Then $\langle n, n_0, \dots, n_{k-1} \rangle$ is α-size. Hence $\langle n, n_0, \dots, n_{k-1} \rangle \in \mathcal{B}^\alpha$ and $n \in \text{base}(\mathcal{B}^\alpha)$, so $\text{base}(\mathcal{B}^\alpha) = M$.

(2) Let $X \subseteq M$ be infinite and let j be least such that $\alpha[X_0, \dots, X_j] = 0$. Then $\langle X_0, \dots, X_j \rangle \in \mathcal{B}^\alpha$ and $\langle X_0, \dots, X_j \rangle \sqsubset X$.

(3) Let $s, t \in \mathcal{B}^\alpha$ and suppose for sake of contradiction $s \sqsubset t$. Then $s \sqsubseteq t^*$ and so it follows that $\alpha[t^*] \leqslant \alpha[s] = 0$, which contradicts the fact that t is α-size.

Assume the system of fundamental sequences is nested. Since a smooth block is automatically a barrier it suffices to prove smoothness. Let $s, t \in \mathcal{B}^\alpha$ be such that $|s| < |t|$. Towards a contradiction assume that for each $i < |s|$, $t(i) \leqslant s(i)$. Since the system of fundamental sequences is nested and $\min t > 1$, s and t^* satisfy the hypothesis of Lemma 4.1.3. Therefore we have that $\alpha[t^*] \leqslant \alpha[s] = 0$, which contradicts the fact that t is α-size. We conclude that $\mathcal{B}^\alpha$ is a smooth barrier. □

The next lemmas provide properties that $\mathcal{B}^\alpha$ inherits from the fixed system of fundamental sequences.

Lemma 4.2.2. *If* $t \in T(\mathcal{B}^\alpha)$ *then* $\text{ht}_{\mathcal{B}^\alpha}(t) = \alpha[t]$.

Proof. We proceed by induction on $\text{ht}_{\mathcal{B}^\alpha}(t)$. If $\text{ht}_{\mathcal{B}^\alpha}(t) = 0$ then $t \in \mathcal{B}^\alpha$, t is α-size by definition and so $\alpha[t] = 0$. If $\text{ht}_{\mathcal{B}^\alpha}(t) > 0$ then $t \in T(\mathcal{B}^\alpha) \setminus \mathcal{B}^\alpha$ and

$$\begin{aligned} \text{ht}_{\mathcal{B}^\alpha}(t) &= \sup_{n \in M} (\text{ht}_{\mathcal{B}^\alpha}(t^\frown\langle n \rangle) + 1) \\ &= \sup_{n \in M} (\alpha[t^\frown\langle n \rangle] + 1) \\ &= \sup_{n \in M} (\alpha[t][n] + 1) = \alpha[t]. \quad \square \end{aligned}$$

Corollary 4.2.3. $\text{ht}(\mathcal{B}^\alpha) = \alpha$.

Lemma 4.2.4. *For each* $t \in T(\mathcal{B}^\alpha)$, $s \in \mathcal{B}^\alpha_t$ *if and only if* $\max t < \min s$ *and* $s \in \mathcal{B}^{\text{ht}_{\mathcal{B}^\alpha}(t)}$.

Proof. If $s \in \mathcal{B}^\alpha_t$ then $\max t < \min s$ and $t^\frown s \in \mathcal{B}^\alpha$ by definition. It follows that $0 = \alpha[t^\frown s] = \alpha[t][s]$ and by Lemma 4.2.2 $\mathrm{ht}_{\mathcal{B}^\alpha}(t)[s] = 0$. In other words, s is $\mathrm{ht}_{\mathcal{B}^\alpha}(t)$-large. Since $t^\frown s$ is α-size, no prefix of s can be $\mathrm{ht}_{\mathcal{B}^\alpha}(t)$-large which means that $s \in \mathcal{B}^{\mathrm{ht}_{\mathcal{B}^\alpha}(t)}$.

Conversely, let $s \in \mathcal{B}^{\mathrm{ht}_{\mathcal{B}^\alpha}(t)}$ with $\max t < \min s$ (so that $t^\frown s$ does make sense). Then, again by Lemma 4.2.2, $\alpha[t^\frown s] = \alpha[t][s] = \mathrm{ht}_{\mathcal{B}^\alpha}(t)[s] = 0$ and so $s \in \mathcal{B}^\alpha_t$. □

Lemma 4.2.5. *Assume the system of fundamental sequences is nested and regular. Let* $\alpha = \beta_0 + \omega^{\beta_1}$ *where* $\beta_0 \gg \omega^{\beta_1}$. *If* $t^\frown s \in \mathcal{B}^\alpha$ *then the following are equivalent:*

(1) $\mathrm{ht}_{\mathcal{B}^\alpha}(t) = \beta_0$,

(2) t *is* ω^{β_1}*-size,*

(3) s *is* β_0*-size.*

Proof. Assuming (1), by Lemma 4.2.4 $s \in \mathcal{B}^{\mathrm{ht}_{\mathcal{B}^\alpha}(t)}$ which means that s is β_0-size, so that we have (3). If (2) fails then either a proper subset or a proper superset of s is β_0-size, contradicting (3). Finally, assuming (2), by Lemma 4.2.2 and regularity of the system of fundamental sequences

$$\mathrm{ht}_{\mathcal{B}^\alpha}(t) = \alpha[t] = (\beta_0 + \omega^{\beta_1})[t] = \beta_0 + (\omega^{\beta_1}[t]) = \beta_0,$$

that is (1). □

Theorem 4.2.6. *Assume the system of fundamental sequences is nested and regular. Then* $\mathcal{B}^\alpha$ *is a* SD*-barrier.*

Proof. Theorem 4.2.1 yields that $\mathcal{B}^\alpha$ is a smooth barrier, so we only need to prove that regularity of the system implies decomposability of the barrier. Let $\omega^{\beta_0} + \ldots + \omega^{\beta_n}$ be the Cantor normal form of α. It is easy to check that regularity of the system of fundamental sequences implies $\mathcal{B}^\alpha = \mathcal{B}^{\omega^{\beta_0}} \oplus \ldots \oplus \mathcal{B}^{\omega^{\beta_n}}$, hence the thesis. □

Next we show how we can obtain fundamental sequences starting from a barrier. We recall that if $\mathcal{B}$ is smooth then each $\mathcal{B}_n$ is smooth.

Lemma 4.2.7. *Let* $\mathcal{B}$ *be a smooth barrier and* $\mathrm{ht}(\mathcal{B}) = \alpha$. *Then* $\alpha[n] = \mathrm{ht}(\mathcal{B}_n)$ *is a fundamental sequence for* α.

Proof. If the sequence is non decreasing there exist $n < m$ with $\alpha[m] < \alpha[n]$, i.e. $\mathrm{ht}(\mathcal{B}_m) < \mathrm{ht}(\mathcal{B}_n)$. Let $\mathcal{B}'_n$ be the restriction of $\mathcal{B}_n$ to $\mathrm{base}(\mathcal{B}_m)$, which is a smooth barrier with the same height as $\mathcal{B}_n$ by Lemma 4.1.17. By Lemma 4.1.18 there exist $s \in \mathcal{B}_m$ and $t \in \mathcal{B}'_n$ such that $s \sqsubset t$. The sequences $\langle m \rangle^\frown s$ and $\langle n \rangle^\frown t$ belong to $\mathcal{B}$ and contradict its smoothness.

Moreover

$$\sup\{\alpha[n]+1 : n \in \omega\} = \sup\{\mathrm{ht}(\mathcal{B}_n)+1 : n \in \omega\} = \mathrm{ht}(\mathcal{B}) = \alpha.$$

Therefore $\alpha[n]$ is a fundamental sequence for α. □

Recall that if T is a well-founded tree then for each $\beta \leqslant \mathrm{ht}(T)$ there exists $s \in T$ such that $\mathrm{ht}_T(s) = \beta$. Using this and the fact that if $\mathcal{B}$ is a smooth barrier then $\mathcal{B}_s$ is a smooth barrier for each $s \in T(\mathcal{B})$, we obtain fundamental sequences for every $\beta \leqslant \mathrm{ht}(\mathcal{B})$. More precisely, for each $s \in T(\mathcal{B}) \setminus \mathcal{B}$ let $\beta_s^{\mathcal{B}} = \mathrm{ht}_{\mathcal{B}}(s) = \mathrm{ht}(\mathcal{B}_s)$: then we define[3]

$$\beta_s^{\mathcal{B}}[n] = \beta^{\mathcal{B}}_{s^\frown\langle m\rangle} \text{ where } m = \min\{k \in \mathrm{base}(\mathcal{B}) : k \geqslant \max(\max s+1, n)\}.$$

Analogously to what we do with systems of fundamental sequences, if $s \in \mathcal{B}$ then we stipulate that $\beta_s^{\mathcal{B}}[n] = 0$ for each n. We will often just write β_s instead of $\beta_s^{\mathcal{B}}$ when the barrier $\mathcal{B}$ is clear from the context. Notice that the fundamental sequence for β_s depends not only on the ordinal β_s but also on the specific $s \in T(\mathcal{B})$: in fact it is easy to provide examples of barriers $\mathcal{B}$ such that for some $s \neq t \in T(\mathcal{B})$ we have $\beta_s = \beta_t$ but $\beta_s[n] \neq \beta_t[n]$ for some n. Therefore we do not have a system of fundamental sequences in the sense of Subsection 1.3.

It is clear that we cannot expect nestedness in the strong sense i.e. that for every $s, t \in T(\mathcal{B})$ and $n > 1$ it does not hold

$$\beta_s > \beta_t > \beta_s[n] > \beta_t[n]$$

(it is easy to build a counterexample). However, it is immediate to check that the following pseudonestedness holds: for each $s, t \in T(\mathcal{B})$ with $s \sqsubset t$ and each $n \geqslant t(|s|)$ we have $\beta_s[n] \geqslant \beta_t$.

SD-barriers enjoy the following version of regularity.

Lemma 4.2.8. *Let $\mathcal{B}$ be a* SD*-barrier of height $\omega^{\beta_0} + \ldots + \omega^{\beta_n}$. Let $\mathcal{B} = \mathcal{B}_0 \oplus \ldots \oplus \mathcal{B}_n$ be a decomposition and let $\mathcal{C} = \mathcal{B}_0 \oplus \ldots \oplus \mathcal{B}_{n-1}$ so that $\mathcal{B} = \mathcal{C} \oplus \mathcal{B}_n$ where $\mathrm{ht}(\mathcal{B}_n) = \omega^{\beta_n}$ and $\gamma = \mathrm{ht}(\mathcal{C}) \ggg \omega^{\beta_n}$. Then, for each m, $\beta^{\mathcal{B}}_{\langle\rangle}[m] = \gamma + \beta^{\mathcal{B}_n}_{\langle\rangle}[m]$.*

Proof. Notice that we just need to prove the statement when $m \in \mathrm{base}(\mathcal{B}) = \mathrm{base}(\mathcal{B}_n)$. For each $s \in \mathcal{B}_n$ we have $\mathrm{ht}_{\mathcal{B}}(s) = \gamma$. Therefore for each $t \in T(\mathcal{B}_n)$ it holds that $\mathrm{ht}_{\mathcal{B}}(t) = \gamma + \mathrm{ht}_{\mathcal{B}_n}(t)$. It follows that $\beta^{\mathcal{B}}_{\langle\rangle}[m] = \mathrm{ht}_{\mathcal{B}}(\langle m\rangle) = \gamma + \mathrm{ht}_{\mathcal{B}_n}(\langle m\rangle) = \gamma + \beta^{\mathcal{B}_n}_{\langle\rangle}[m]$. □

[3] the definition is a bit involved because we need to define $\beta_s^{\mathcal{B}}[n]$ even when $n \notin \mathrm{base}(\mathcal{B})$.

4.3 RAMSEY THEOREM FOR BARRIERS

By a k-*coloring* of a set X we mean a function $c\colon X \to \{0,\dots,k-1\}$. Typically X consists of subsets of some finite set s. A *homogeneous set* for c is a set $t \subseteq s$ for which there exists $i < k$ (the *color of* t) such that c colors all subsets of t which belong to X with color i.

The arrow notation was introduced by Erdös and Rado in [ER53]. For each $m, n, k, N \in \mathbb{N}$ with $m \geqslant n$ we write

$$N \to (m)^n_k$$

to mean that for each $s \in [\mathbb{N}]^N$ and each k-coloring of $[s]^n$ there is a homogeneous subset of s of cardinality m.

We adapt the arrow notation to our framework. If $\mathcal{C}$ is a front let us denote by $[s]^{\mathcal{C}}$ the collection of all $\mathcal{C}$-size subsets of s.

Definition 4.3.1. Let $k \in \mathbb{N}$ and let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be fronts where $\mathrm{base}(\mathcal{B}) = \mathrm{base}(\mathcal{A})$ is a final segment of $\mathrm{base}(\mathcal{C})$. We write

$$\mathcal{B} \to (\mathcal{A})^{\mathcal{C}}_k$$

to mean that for each $(\mathbb{1} \oplus \mathcal{B})$-size s and each k-coloring of $[s]^{\mathcal{C}}$, there exists a homogeneous $(\mathcal{C} \oplus \mathcal{A})$-size subset of s.

We remark that we ask that the homogeneous set is $(\mathcal{C} \oplus \mathcal{A})$-size instead of just $\mathcal{A}$-size to be sure that there are actually $\mathcal{C}$-size sets to be colored. We want to avoid the case in which the set is trivially homogeneous just because it does not have $\mathcal{C}$-size subsets. Similarly the use of $(\mathbb{1} \oplus \mathcal{B})$-size sets makes the statements and the arguments more natural.

As usual in Ramsey theory, the following asymmetric definition is useful.

Definition 4.3.2. Let $k \in \mathbb{N}$ and let $\mathcal{A}_0, \dots, \mathcal{A}_{k-1}$, $\mathcal{B}$ and $\mathcal{C}$ be fronts where $\mathrm{base}(\mathcal{B}) = \bigcap_{i<k} \mathrm{base}(\mathcal{A}_i)$ and each $\mathrm{base}(\mathcal{A}_i)$ is a final segment of $\mathrm{base}(\mathcal{C})$. We write

$$\mathcal{B} \to (\mathcal{A}_0, \dots, \mathcal{A}_{k-1})^{\mathcal{C}}_k$$

to mean that for each $(\mathbb{1} \oplus \mathcal{B})$-size s and each k-coloring of $[s]^{\mathcal{C}}$, there exists a homogeneous $(\mathcal{C} \oplus \mathcal{A}_i)$-size subset of s of color i for some $i < k$.

If $\mathcal{A}$ and $\mathcal{C}$ are fronts such that $\mathrm{base}(\mathcal{A})$ is a final segment of $\mathrm{base}(\mathcal{C})$, we say that a k-coloring $c\colon [s]^{\mathcal{C}} \to k$ is *bad* (relative to $\mathcal{C}$ and $\mathcal{A}$) if it does not contain a homogeneous $(\mathcal{C} \oplus \mathcal{A})$-size set. More generally, if $\mathcal{A}_0, \dots, \mathcal{A}_{k-1}$ are fronts such that each of $\mathrm{base}(\mathcal{A}_i)$ is a final segment of $\mathrm{base}(\mathcal{C})$, we say that a k-coloring $c\colon [s]^{\mathcal{C}} \to k$ is *bad* (relative to $\mathcal{C}, \mathcal{A}_0, \dots, \mathcal{A}_{k-1}$) if it does not contain a homogeneous $(\mathcal{C} \oplus \mathcal{A}_i)$-size set of color i for any $i < k$.

Lemma 4.3.3. *Let* $k \in \mathbb{N}$ *and let* $\mathcal{A}_0, \dots, \mathcal{A}_{k-1}, \mathcal{C}$ *be fronts such that each of* $\mathrm{base}(\mathcal{A}_i)$ *is a final segment of* $\mathrm{base}(\mathcal{C})$. *Then every infinite* $Y \subseteq \bigcap_{i<k} \mathrm{base}(\mathcal{A}_i)$ *has an initial segment* s_Y *such that for each coloring* $c\colon [s_Y]^{\mathcal{C}} \to k$ *there exists a homogeneous* $(\mathcal{C} \oplus \mathcal{A}_i)$*-size subset of* s_Y *of color* i *for some* $i < k$.

Proof. Nash-Williams proved in [Nas65] that each k-coloring c of $[Y]^{\mathcal{C}}$ has an infinite homogeneous set. This is essentially the clopen Ramsey theorem [GP73]. A standard compactness argument now completes the proof. □

Notice that in the previous lemma, if at least one between $\mathcal{A}$ and $\mathcal{C}$ is a block, then s_Y cannot be $\langle\rangle$.

We can now obtain what we call the barrier Ramsey theorem[4], which allows us to introduce the Ramsey ordinals in our framework.

Theorem 4.3.4 (Barrier Ramsey theorem)**.** *Let* $k \in \mathbb{N}$ *and let* $\mathcal{A}$ *and* $\mathcal{C}$ *be* SD*-barriers such that* $\mathrm{base}(\mathcal{A})$ *is a final segment of* $\mathrm{base}(\mathcal{C})$. *Then there exists a* SD*-barrier* $\mathcal{B}$ *with* $\mathrm{base}(\mathcal{B}) = \mathrm{base}(\mathcal{A})$ *such that* $\mathcal{B} \to (\mathcal{A})^{\mathcal{C}}_k$.

Proof. If $\mathcal{A}$ is the degenerate front then $\mathcal{B} = \mathcal{C}$ works. If $\mathcal{C}$ is the degenerate front then $\mathcal{B} = \mathcal{A}$ works. Assume that neither $\mathcal{A}$ nor $\mathcal{C}$ is the degenerate front. In the notation of Lemma 4.3.3, let $\mathcal{A}_i = \mathcal{A}$ for each $i < k$ and consider

$$\mathcal{Y} = \{s \in [\mathrm{base}(\mathcal{A})]^{<\omega} : s = s_Y \text{ for some infinite } Y \subseteq \mathrm{base}(\mathcal{A})\}.$$

Then $\mathcal{B}' = \{s \in \mathcal{Y} : \forall t \sqsubset s (t \notin \mathcal{Y})\}$ is a block with the same base as $\mathcal{A}$ and satisfies $\mathcal{B}' \to (\mathcal{A})^{\mathcal{C}}_k$. By Lemma 1.4.2 and Corollary 4.1.26 starting from the block $\mathcal{B}'$ we can build a SD-barrier $\mathcal{B}$ with $\mathrm{base}(\mathcal{B}) = \mathrm{base}(\mathcal{B}')$ and $\mathrm{ht}(\mathcal{B}) = \mathrm{ht}(\mathcal{B}')$ such that each element of $\mathcal{B}$ is $\mathcal{B}'$-large. Therefore $\mathcal{B}$ is a SD-barrier with the same base as $\mathcal{A}$ that satisfies $\mathcal{B} \to (\mathcal{A})^{\mathcal{C}}_k$. □

We aim to relate $\mathrm{ht}(\mathcal{B})$ for $\mathcal{B}$ satisfying Theorem 4.3.4 to $\mathrm{ht}(\mathcal{A})$ and $\mathrm{ht}(\mathcal{C})$.

Definition 4.3.5. For $k \in \mathbb{N}$ and countable ordinals α and γ, let $\mathrm{Ram}(\alpha)^{\gamma}_k$ be the least ordinal β such that for all SD-barriers $\mathcal{A}$ and $\mathcal{C}$ with $\mathrm{ht}(\mathcal{A}) = \alpha$, $\mathrm{ht}(\mathcal{C}) = \gamma$ and $\mathrm{base}(\mathcal{A})$ a final segment of $\mathrm{base}(\mathcal{C})$, there exists a SD-barrier $\mathcal{B}$ with $\mathrm{ht}(\mathcal{B}) = \beta$ and $\mathrm{base}(\mathcal{B}) = \mathrm{base}(\mathcal{A})$ such that $\mathcal{B} \to (\mathcal{A})^{\mathcal{C}}_k$. Moreover

$$\mathrm{Ram}(\alpha)^{\gamma}_{<\omega} = \sup_{k \in \mathbb{N}} \mathrm{Ram}(\alpha)^{\gamma}_k.$$

Again, we extend our definition to the asymmetric case.

[4] beware that the terminology barrier Ramsey theorem is used in [Car+24] for a different statement.

Definition 4.3.6. For $k \in \mathbb{N}$ and countable ordinals $\alpha_0, \dots, \alpha_{k-1}, \gamma$, let $\mathrm{Ram}(\alpha_0, \dots, \alpha_{k-1})^\gamma_k$ be the least ordinal β such that for all SD-barriers $\mathcal{A}_0, \dots, \mathcal{A}_{k-1}, \mathcal{C}$ with $\mathrm{ht}(\mathcal{A}_i) = \alpha_i$, $\mathrm{ht}(\mathcal{C}) = \gamma$ and each $\mathrm{base}(\mathcal{A}_i)$ a final segment of $\mathrm{base}(\mathcal{C})$, there exists a SD-barrier $\mathcal{B}$ with $\mathrm{ht}(\mathcal{B}) = \beta$ and $\mathrm{base}(\mathcal{B}) = \bigcap_{i<k} \mathrm{base}(\mathcal{A}_i)$ such that $\mathcal{B} \to (\mathcal{A}_0, \dots, \mathcal{A}_{k-1})^{\mathcal{C}}_k$. Moreover

$$\mathrm{Ram}(\alpha_0, \dots, \alpha_{k-1})^\gamma_{<\omega} = \sup_{k \in \mathbb{N}} \mathrm{Ram}(\alpha_0, \dots, \alpha_{k-1})^\gamma_k.$$

Before stating our main theorem, we need to introduce a logarithm function related to the Veblen hierarchy.

Definition 4.3.7. Let $\gamma = \omega^{\delta_0} + \dots + \omega^{\delta_n}$ be an ordinal written in Cantor normal form. We define the function $\varphi_{\log \gamma}$ as the composition $\varphi_{\delta_0} \circ \dots \circ \varphi_{\delta_n}$. Moreover let $\varphi_{\log 0}$ be the identity function.

Fernández-Duque and Joosten introduced and studied the concept of hyperations of normal functions in [FJ13]. Our logarithm operator is the particular case of the hyperation of ordinal exponentiation in base ω, as noticed in [FJ13, Corollary 4.10][5].

We are now ready to state the main result of the paper.

Theorem 4.3.8 (Main theorem). *For any countable ordinals α and γ such that $\gamma < \alpha$ and $\alpha \geqslant \omega$ we have* $\mathrm{Ram}(\alpha)^{1+\gamma}_{<\omega} = \varphi_{\log \gamma}(\alpha \cdot \omega)$.

The appearance of $1 + \gamma$ in the above statement allows us to avoid using different formulas in the finite and infinite case. The hypothesis that α is infinite is necessary: indeed, when α (and hence γ) is finite, the finite Ramsey theorem always yields $\mathrm{Ram}(\alpha)^{1+\gamma}_{<\omega} = \omega$. Finally, notice that we do not get sharp Ramsey ordinals for a fixed number k of colors. This is not surprising, as sharp Ramsey ordinals are known only in very few finite cases.

4.4 THE BARRIER PIGEONHOLE PRINCIPLE

We start by studying the Barrier Pigeonhole Principle and proving the case $\gamma = 0$ of Theorem 4.3.8. We use $\#$ and $\otimes$ to indicate respectively the natural (or Hessenberg) ordinal sum and product. The following observation about $\#$ is useful.

Proposition 4.4.1. *Let α and β be ordinals such that $\alpha = \sup_{n \in \mathbb{N}} \alpha_n$ and $\beta = \sup_{n \in \mathbb{N}} \beta_n$. Then* $\alpha \# \beta = \max\left(\sup_{n \in \mathbb{N}}(\alpha_n \# \beta), \sup_{n \in \mathbb{N}}(\alpha \# \beta_n)\right)$.

[5] we thank Fedor Pakhomov for pointing us to [FJ13].

Definition 4.4.2. Let $\mathcal{A}_0, \ldots, \mathcal{A}_{k-1}$ be fronts and Y a set such that each $\mathrm{base}(\mathcal{A}_i)$ is a final segment of Y and let $X = \bigcap_{i<k} \mathrm{base}(\mathcal{A}_i)$. Then we define

$$T(\mathcal{B}) = \{s \in [X]^{<\omega} : s \text{ has a bad } k\text{-coloring relative to } \mathbb{1}, \mathcal{A}_0, \ldots, \mathcal{A}_{k-1}\}.$$

Let $\mathcal{B}$ be the set of leaves of $T(\mathcal{B})$ (justifying the name of the initial set).

We sometimes write $T(\mathcal{B}(\mathcal{A}_0, \ldots \mathcal{A}_{k-1}))$ and $\mathcal{B}(\mathcal{A}_0, \ldots \mathcal{A}_{k-1})$ to highlight the dependence on the fronts $\mathcal{A}_0, \ldots, \mathcal{A}_{k-1}$.

Lemma 4.4.3. *Let $\mathcal{A}_0, \ldots, \mathcal{A}_{k-1}$ be smooth barriers and Y a set such that each $\mathrm{base}(\mathcal{A}_i)$ is a final segment of Y. Then $\mathcal{B} = \mathcal{B}(\mathcal{A}_0, \ldots \mathcal{A}_{k-1})$ of Definition 4.4.2 is a smooth barrier.*

Proof. By Lemma 4.3.3 $T(\mathcal{B})$ is a well founded tree and hence $\mathcal{B}$ is a front with base $X = \bigcap_{i<k} \mathrm{base}(\mathcal{A}_i)$. We only need to prove that $\mathcal{B}$ is smooth. Towards a contradiction, let $s, t \in \mathcal{B}$ with $|s| < |t|$ be such that $t(i) \leqslant s(i)$ for all $i < |s|$ and let $m > \max(s \cup t)$. Each k-coloring of either $s^\frown\langle m\rangle$ or $t^\frown\langle m\rangle$ has a homogeneous $(\mathbb{1} \oplus \mathcal{A}_i)$-size subset of color i for some $i < k$. Let $c\colon t \to k$ be a bad k-coloring and let $\bar{c}\colon s^\frown\langle m\rangle \to k$ be defined as $\bar{c}(s^\frown\langle m\rangle(j)) = c(t(j))$ for all $j < |s| + 1 \leqslant |t|$. There exists a homogeneous $(\mathbb{1} \oplus \mathcal{A}_{i_0})$-size set v_s for $\bar{c}$ of color $i_0 < k$. Extend c to $t^\frown\langle m\rangle$ by setting $c(m) = i_0$. Then the set $v_t = \{n \in t^\frown\langle m\rangle : c(n) = i_0\}$ is $(\mathbb{1} \oplus \mathcal{A}_{i_0})$-size and $|v_t| \geqslant |v_s| + 1$. For each $\ell < |v_s|$, either $v_s(\ell) = s(j)$ for some $j < |s|$ and $v_t(\ell) = t(j)$ or $v_s(\ell) = m$ and $v_t(\ell) = t(|s|)$. Therefore for each $\ell < |v_s|$, $v_t(\ell) \leqslant v_s(\ell)$, contradicting the smoothness of the barrier $\mathbb{1} \oplus \mathcal{A}_{i_0}$. □

Lemma 4.4.4. *Let $\mathcal{A}_0, \ldots, \mathcal{A}_{k-1}$ be smooth barriers and Y a set such that each $\mathrm{base}(\mathcal{A}_i)$ is a final segment of Y. Then*

$$\mathrm{ht}(\mathcal{B}(\mathcal{A}_0, \ldots, \mathcal{A}_{k-1})) = \mathrm{ht}(\mathcal{A}_0) \# \ldots \# \mathrm{ht}(\mathcal{A}_{k-1}).$$

Proof. Denote $\mathrm{ht}(\mathcal{A}_i)$ by α_i, $\mathcal{B}(\mathcal{A}_0, \ldots, \mathcal{A}_{k-1})$ by $\mathcal{B}$ and $\mathrm{base}(\mathcal{B})$ by X. We proceed by induction on the finite tuples of ordinals $(\alpha_0, \ldots, \alpha_{k-1})$ using the well founded partial order defined by

$$(\gamma_0, \ldots, \gamma_{\ell-1}) \leqslant (\delta_0, \ldots, \delta_{\ell'-1}) \text{ iff } \ell < \ell' \vee (\ell = \ell' \wedge \forall i < \ell\, \gamma_i \leqslant \delta_i).$$

If $(\alpha_0, \ldots, \alpha_{k-1}) = (0, \ldots, 0)$ then every $\mathcal{A}_i$ is the degenerate front and so $\mathcal{B}(\mathcal{A}_0, \ldots, \mathcal{A}_{k-1})$ is the degenerate front. Hence $\mathrm{ht}(\mathcal{B}(\mathcal{A}_0, \ldots, \mathcal{A}_{k-1})) = 0$ as required.

If for some $i < k$ we have $\alpha_i = 0$ (that is, $\mathcal{A}_i$ is the degenerate front) then each singleton is $(\mathbb{1} \oplus \mathcal{A}_i)$-size and this means that a bad k-coloring does not use color i. Hence $\mathcal{B}(\mathcal{A}_0, \ldots, \mathcal{A}_{k-1}) = \mathcal{B}(\mathcal{A}_0, \ldots, \mathcal{A}_{i-1}, \mathcal{A}_{i+1}, \ldots, \mathcal{A}_{k-1})$ and we can use the induction hypothesis.

If for every $i < k$ we have $\alpha_i > 0$ then we first show that for each $n \in X$

$$T(\mathcal{B}_n) = \bigcup_{j<k} T(\mathcal{B}(\mathcal{A}_0, \dots, (\mathcal{A}_j)_n, \dots, \mathcal{A}_{k-1})).$$

Let $s \in T(\mathcal{B}_n)$ namely $\langle n \rangle^\frown s \in T(\mathcal{B}(\mathcal{A}_0, \dots, \mathcal{A}_{k-1}))$. There exists a bad k-coloring c of $\langle n \rangle^\frown s$. If $c(n) = j$ then $\langle n \rangle^\frown \{m \in s : c(m) = j\}$ belongs to $T(\mathcal{A}_j)$ and so $s \in T(\mathcal{B}(\mathcal{A}_0, \dots, (\mathcal{A}_j)_n, \dots, \mathcal{A}_{k-1}))$. Conversely, let $s \in T(\mathcal{B}(\mathcal{A}_0, \dots, (\mathcal{A}_j)_n, \dots, \mathcal{A}_{k-1}))$ for some $j < k$. There exists a bad k-coloring c of s and in particular the set $\langle n \rangle^\frown \{m \in s : c(m) = j\}$ belongs to $T(\mathcal{A}_j)$. Recall that the base of $\mathcal{B}(\mathcal{A}_0, \dots, (\mathcal{A}_j)_n, \dots, \mathcal{A}_{k-1})$ is $X \setminus \{0, \dots, n\}$. Therefore $\langle n \rangle^\frown s \in [X]^{<\omega}$ and $\langle n \rangle^\frown s \in T(\mathcal{B})$ which means that $s \in T(\mathcal{B}_n)$.

It follows that for each $n \in X$

$$\mathcal{B}_n = \bigsqcup_{j<k} \mathcal{B}(\mathcal{A}_0, \dots, (\mathcal{A}_j)_n, \dots, \mathcal{A}_{k-1}).$$

We are now ready to compute $ht(\mathcal{B})$. For each $n \in X$

$$\begin{aligned} ht_{\mathcal{B}}(\langle n \rangle) &= ht\left(\bigsqcup_{j<k} \mathcal{B}(\mathcal{A}_0, \dots, (\mathcal{A}_j)_n, \dots, \mathcal{A}_{k-1})\right) \\ &= \max_{j<k}(ht(\mathcal{B}(\mathcal{A}_0, \dots, (\mathcal{A}_j)_n, \dots, \mathcal{A}_{k-1})) \end{aligned}$$

where the last equality follows by Remark 1.4.4. By inductive hypothesis $ht_{\mathcal{B}}(\langle n \rangle) = \max_{j<k}(\alpha_0 \# \dots \# ht((\mathcal{A}_j)_n) \# \dots \# \alpha_{k-1})$. Hence, using commutativity of the natural sum and Proposition 4.4.1, we obtain

$$\begin{aligned} ht(\mathcal{B}) &= \sup_{n \in X}(\max_{j<k}(\alpha_0 \# \dots \# ht((\mathcal{A}_j)_n) \# \dots \# \alpha_{k-1}) + 1) \\ &= \alpha_0 \# \dots \# \alpha_{k-1}. \quad \square \end{aligned}$$

Theorem 4.4.5. *For all countable ordinals* $\alpha_0, \dots, \alpha_{k-1}$

$$Ram(\alpha_0, \dots, \alpha_{k-1})^1_k = \alpha_0 \# \dots \# \alpha_{k-1}.$$

Proof. The unique SD-barrier of height 1 is the singleton barrier $\mathbb{1}$ and we may assume its base is $\mathbb{N}$. Let $\mathcal{A}_0, \dots, \mathcal{A}_{k-1}$ be SD-barriers such that each base$(\mathcal{A}_i)$ is a final segment of $\mathbb{N}$ and with height respectively $\alpha_0, \dots, \alpha_{k-1}$. Let $\mathcal{B}$ be the set $\mathcal{B}(\mathcal{A}_0, \dots, \mathcal{A}_{k-1})$ of Definition 4.4.2. By Lemma 4.4.3 $\mathcal{B}$ is a smooth barrier with base $X = \bigcap_{i<k} \text{base}(\mathcal{A}_i)$ and by definition $\mathcal{B} \to (\mathcal{A}_0, \dots, \mathcal{A}_{k-1})^{\mathbb{1}}_k$. Moreover, by Lemma 4.4.4, $ht(\mathcal{B}) = \alpha_0 \# \dots \# \alpha_{k-1}$. By Corollary

4.1.26 there exists a SD-barrier with same base and same height as $\mathcal{B}$ and such that all of its elements are $\mathcal{B}$-large. Such SD-barrier shows that $\mathrm{Ram}(\alpha_0,\dots,\alpha_{k-1})^1_k \leqslant \alpha_0 \# \dots \# \alpha_{k-1}$.

For the converse inequality fix SD-barriers $\mathcal{A}_0,\dots,\mathcal{A}_{k-1}$ with $\mathrm{base}(\mathcal{A}_i) = \mathbb{N}$ and $\mathrm{ht}(\mathcal{A}_i) = \alpha_i$. We need to prove that if $\mathcal{D}$ is a SD-barrier such that $\mathrm{base}(\mathcal{D}) = \mathbb{N}$ and $\mathrm{ht}(\mathcal{D}) < \alpha_0 \# \dots \# \alpha_{k-1}$, then $\mathcal{D} \nrightarrow (\mathcal{A}_0,\dots,\mathcal{A}_{k-1})^{1}_k$. Thus we need to find $s \in \mathcal{D}$ and $n \in \mathbb{N}$ with $n > \max s$ such that $s^\frown\langle n\rangle$ has a bad k-coloring. By Lemma 4.1.18 there are $s \in \mathcal{D}$ and $t \in \mathcal{B}(\mathcal{A}_0,\dots,\mathcal{A}_{k-1})$ such that $s \sqsubset t$. By definition of $\mathcal{B}(\mathcal{A}_0,\dots,\mathcal{A}_{k-1})$, t has a bad k-coloring and its restriction to $s^\frown\langle t(|s|)\rangle$ is a bad k-coloring. □

Corollary 4.4.6 (Barrier Pigeonhole Principle)**.** $\mathrm{Ram}(\alpha)^1_k = \alpha \otimes k$ *and therefore* $\mathrm{Ram}(\alpha)^1_{<\omega} = \alpha \cdot \omega$.

Proof. For the first part apply Theorem 4.4.5 to the case $\alpha_0 = \dots = \alpha_{k-1} = \alpha$. For the second part it suffices to notice that $\sup_{k\in\mathbb{N}}(\alpha \otimes k) = \alpha \cdot \omega$. □

4.5 THE LOWER BOUND

In this section we establish the lower bound for Theorem 4.3.8, i.e.

$$\mathrm{Ram}(\alpha)^{1+\gamma}_{<\omega} \geqslant \varphi_{\log\gamma}(\alpha \cdot \omega).$$

To prove this we use the nested and regular system of fundamental sequences on Γ_ζ introduced in Definition 4.1.13, and the largeness notions associated to this system.

We stress that, unlike the upper bound that we prove in Section 4.6, the lower bound is obtained for the supremum over $k \in \mathbb{N}$ of ordinal Ramsey numbers for k-colorings rather than for fixed values of k.

The next theorem is the main result of the section.

Theorem 4.5.1. *Let* $k > 0$, $\gamma < \alpha < \Gamma_\zeta$, $\alpha \geqslant \omega$ *and* $\mu < \varphi_{\log\gamma}(\alpha \otimes k)$. *There exists an infinite set* M *such that for each* $s \in [M]^{1\uplus\mu}$ *there is a coloring* $c\colon [s]^{1+\gamma} \to k+3$ *with no homogeneous* $((1+\gamma)\uplus\alpha)$*-size subsets.*

We now show how Theorem 4.5.1 yields our lower bound. Fix $\gamma < \alpha < \Gamma_\zeta$ and $\nu < \varphi_{\log\gamma}(\alpha \otimes \omega) < \Gamma_\zeta$ ordinals. Pick $k \in \mathbb{N}$ and μ an ordinal such that $\nu < \mu < \varphi_{\log\gamma}(\alpha \otimes k)$. Apply Theorem 4.5.1 to k, γ, α, μ to get an infinite set M (we may assume $\min M > 1$). Now let $\mathcal{A} = [M]^\alpha$, $\mathcal{C} = [M]^{1+\gamma}$ and $\mathcal{D} = [M]^{1\uplus\mu}$. By Theorem 4.2.6 $\mathcal{A}$, $\mathcal{C}$ and $\mathcal{D}$ are SD-barrier. Let $\mathcal{B}$ be any SD-barrier of height ν with $\mathrm{base}(\mathcal{B}) = M$. Since $\mathrm{ht}(\mathcal{B}) = \nu < 1+\mu = \mathrm{ht}(\mathcal{D})$, by Lemma 4.1.18 there are $s \in \mathcal{B}$ and $t \in \mathcal{D}$ such that $s \sqsubset t$. Then $s^\frown\langle t(|s|)\rangle$ inherits a bad coloring of its $(1+\gamma)$-size sets with $k+3$ colors from t and from the fact that $t \in \mathcal{D}$. This means that $\mathcal{B} \not\to (\mathcal{A})^{\mathcal{C}}_{k+3}$ and, since $\mathcal{B}$ is a generic SD-barrier of height ν with $\mathrm{base}(\mathcal{B}) = \mathrm{base}(\mathcal{A})$, we conclude.

Therefore we only need to prove Theorem 4.5.1. For the rest of the section, fix α, γ, μ and k as in the statement of the theorem. We point out that the strategy we adopt for the lower bound is close to the proof strategy of [LN92] to establish lower bounds on the Paris-Harrington principle using colorings based on comparisons of ordinal terms.

We need to introduce some machinery. We start defining a coloring on tuples of ordinals and the first step is to define the *overline function*.

Definition 4.5.2. Given ordinals $\beta > \delta$, let $\overline{\beta, \delta}$ be the largest exponent in the Cantor normal form of β which differs from the corresponding term in the Cantor normal form of δ. That is, if the Cantor normal forms are

$$\beta = \omega^{\beta_0} + \ldots + \omega^{\beta_l} + \omega^{\beta_{l+1}} + \ldots + \omega^{\beta_n}$$
$$\delta = \omega^{\beta_0} + \ldots + \omega^{\beta_l} + \omega^{\delta_{l+1}} + \ldots + \omega^{\delta_m}$$

and $\beta_{l+1} > \delta_{l+1}$, then $\overline{\beta, \delta} = \beta_{l+1}$. If $\beta \leqslant \delta$, we let $\overline{\beta, \delta} = 0$.

We now need to climb the Veblen hierarchy and for that we introduce the so called *peeling functions*, which are length preserving functions on finite sequences of ordinals. We denote finite sequences of ordinals with upper case letters from the beginning of the alphabet. We reserve as usual lower case letters from the second half of the alphabet for finite sequences of numbers.

We also introduce the concept of the collection of subterms of an ordinal. To do that we use the notion of multiset, i.e. a set in which elements can occur multiple (though finitely many) times. Recall that if A and B are multisets then $A + B$ is the multiset where each element has multiplicity the sum of its multiplicities in A and B.

Definition 4.5.3. We recursively define the multiset $\mathrm{Sub}(\beta)$ of subterms of an ordinal β as follows.

- If $\beta = 0$, let $\mathrm{Sub}(\beta) = \{0\}$.
- If $\beta = \varphi_{\beta_0}(\beta_1)$ with $\beta_1 < \beta$, let $\mathrm{Sub}(\beta) = \{\beta\} + \mathrm{Sub}(\beta_1)$.
- If $\beta = \omega^{\beta_0} + \ldots + \omega^{\beta_n}$ with $n > 0$ and $\beta_0 \geqslant \ldots \geqslant \beta_n$, let $\mathrm{Sub}(\beta) = \{\beta\} + \mathrm{Sub}(\omega^{\beta_0}) + \ldots + \mathrm{Sub}(\omega^{\beta_n})$.

Definition 4.5.4. We define functions $\bar{p}_\delta : \mathrm{ON}^{<\omega} \to \mathrm{ON}^{<\omega}$ by induction on δ as follows. Let $\bar{p}_0$ be the identity function, and let

$$\bar{p}_1(\beta_0, \beta_1, \dots, \beta_l) = (\overline{\beta_0, \beta_1}, \overline{\beta_1, \beta_2}, \dots, \overline{\beta_k, 0}).$$

For ordinals of the form $\rho + \omega^\delta$ with $\rho \gg \omega^\delta$, we let

$$\bar{p}_{\rho+\omega^\delta} = \bar{p}_{\omega^\delta} \circ \bar{p}_\rho.$$

On infinite ordinals of the form ω^δ, we first define

$$\bar{p}_{<\omega^\delta}(A) = \lim_{\rho \to \omega^\delta} \bar{p}_\rho(A).$$

We prove in Lemma 4.5.5 below that for each ordinal $\nu \leqslant \rho$ and each $i < |A|$, $\bar{p}_\rho(A) \in \mathrm{Sub}(\bar{p}_\nu(A))$. Therefore, $\bar{p}_\rho(A)$ is coordinate wise non increasing as a function of ρ, and the limit above exists. Furthermore, we see in Lemma 4.5.5 that each ordinal in the tuple $\bar{p}_{<\omega^\delta}(A)$ is either 0 or a fixed point of $\varphi_{\delta'}$ for every $\delta' < \delta$ and thus belongs to the image of φ_δ. Then we define $\bar{p}_{\omega^\delta}(A)$ by "peeling off"one application of φ_δ from each nonzero entry in $\bar{p}_{<\omega^\delta}(A)$. In other words

$$\bar{p}_{\omega^\delta}(A) = \varphi_\delta^{-1} \circ \bar{p}_{<\omega^\delta}(A)$$

where we are applying φ_δ^{-1} to each nonzero entry of $\bar{p}_{<\omega^\delta}(A)$.

We write $p_\delta(A)$ (without the bar) for the first element of $\bar{p}_\delta(A)$.

Notice that to be consistent with the terminology we may actually say that $\bar{p}_1$ peels off one application of φ_0. This is appropriate since $\bar{p}_1 = \bar{p}_{\omega^0}$.

Lemma 4.5.5. *The following properties of the peeling functions hold.*

(1) For every ρ and $\nu < \rho$ we have that $\bar{p}_\rho(A)(i) \in \mathrm{Sub}(\bar{p}_\nu(A)(i))$ for each $i < |A|$.

(2) Each entry of $\bar{p}_{<\omega^\delta}(A)$ is either 0 or a fixed point of $\varphi_{\delta'}$ for every $\delta' < \delta$.

(3) If $A(i) < \varphi_\delta(\beta)$ for all $i < |A|$, then for all $i < |A|$ such that $\bar{p}_{\omega^\delta}(A)(i) > 0$ we have $\bar{p}_{\omega^\delta}(A)(i) < \beta$.

(4) If $A(i) < \varphi_{\log \delta}(\beta)$ for all $i < |A|$, then for all $i < |A|$ such that $\bar{p}_\delta(A)(i) > 0$ we have $\bar{p}_\delta(A)(i) < \beta$.

The conclusions of (3) and (4) are stated as implications to include the case $\beta = 0$.

Proof. We prove (1) by induction on ρ. For $\rho = 0$ the statement is trivial, while for $\rho = 1$ it suffices to notice that $\bar{p}_1$ outputs for each entry either 0 or the exponent of one of its Cantor normal form terms, which is a subterm of the corresponding entry of $A = \bar{p}_0(A)$.

When $\rho = \omega^\delta$, we know by inductive hypothesis that for each $\nu < \mu < \omega^\delta$, each entry of $\bar{p}_\mu(A)$ is a subterm of the corresponding entry of $\bar{p}_\nu(A)$. Hence the limit $\bar{p}_{<\omega^\delta}(A)$ is well defined, and each of its entries is a subterm of the corresponding entry of $\bar{p}_\nu(A)$ for each $\nu < \omega^\delta$. By definition of $\bar{p}_{\omega^\delta}$ we peel off one application of φ_δ from each nonzero entry of $\bar{p}_{<\omega^\delta}(A)$. It follows that each entry of $\bar{p}_{\omega^\delta}(A)$ is a subterm of the corresponding entry of $\bar{p}_{<\omega^\delta}(A)$ and so it is also a subterm of the corresponding entry of $\bar{p}_\nu(A)$ for each $\nu < \omega^\delta$ as required.

Finally, consider an ordinal of the form $\rho = \mu + \omega^\delta$ for $\mu \ggg \omega^\delta > 1$ and let $\nu < \mu + \omega^\delta$. In this case $\bar{p}_{\mu+\omega^\delta}(A) = \bar{p}_{\omega^\delta}(\bar{p}_\mu(A))$ by definition. If $\nu \leqslant \mu$ then by inductive hypothesis each entry of $\bar{p}_\mu(A)$ is a subterm of the corresponding entry of $\bar{p}_\nu(A)$. Then since $\omega^\delta < \mu + \omega^\delta$ again by inductive hypothesis we know that each entry of $\bar{p}_{\omega^\delta}(\bar{p}_\mu(A))$ is a subterm of the corresponding entry of $\bar{p}_\mu(A)$. We are left with the case $\mu < \nu < \mu + \omega^\delta$ which means that $\nu = \mu + \sigma$ for some σ such that $\mu \ggg \sigma$ and $0 < \sigma < \omega^\delta$. Then $\bar{p}_\nu(A) = \bar{p}_\sigma(\bar{p}_\mu(A))$ by definition. Since $\sigma < \omega^\delta < \mu + \omega^\delta$ by inductive hypothesis we know that each entry of $\bar{p}_{\omega^\delta}(\bar{p}_\mu(A))$ is a subterm of the corresponding entry of $\bar{p}_\sigma(\bar{p}_\mu(A))$.

For (2) we first claim that $\bar{p}_{<\omega^\delta}(A)$ is a fixed point of $\bar{p}_{\omega^{\delta'}}$ for all $\delta' < \delta$. Let $\rho < \omega^\delta$ such that each entry in $\bar{p}_{<\omega^\delta}(A)$ has stabilized at $\bar{p}_\rho(A)$ (ρ exists by (1)) and pick ρ' with $\rho \leqslant \rho' < \omega^\delta$ such that $\rho' \ggg \omega^{\delta'}$. Then, we have that

$$\bar{p}_{\omega^{\delta'}}(\bar{p}_{<\omega^\delta}(A)) = \bar{p}_{\omega^{\delta'}}(\bar{p}_{\rho'}(A)) = \bar{p}_{\rho'+\omega^{\delta'}}(A) = \bar{p}_{<\omega^\delta}(A).$$

This completes the proof of the claim.

Considering the case $\delta' = 0$ we see that $\bar{p}_{<\omega^\delta}(A)$ is a fixed point of $\bar{p}_1$, and this can happen only if the Cantor normal form of each entry of the ordinals in $\bar{p}_{<\omega^\delta}(A)$ contains only one term and that term is either 0 or a fixed point of the exponential function φ_0. Considering larger δ', we see that no entry of $\bar{p}_{<\omega^\delta}(A)$ can be of the form $\varphi_{\delta'}(\beta)$ with $\beta < \varphi_{\delta'}(\beta)$, as otherwise that last application of $\varphi_{\delta'}$ would have been peeled off at some previous stage. To see this notice that for each n, $\omega^{\delta'} \cdot n < \omega^\delta$ and each time you reach $\bar{p}_{\omega^{\delta'} \cdot n}$ in the recursive definition of the peeling functions, at least n applications of $\varphi_{\delta'}$ are peeled off. Therefore, before stage ω^δ there will be no more applications of φ'_δ. All the entries that did not go all the way down to 0 must then be fixed points of $\varphi_{\delta'}$ for all $\delta' < \delta$ and part (2) is proved.

Part (3) follows from the second, as the only non zero ordinals smaller than $\varphi_\delta(\beta)$ that are invariant under the operations $\varphi_{\delta'}$ for all $\delta' < \delta$ are those of the form $\varphi_\delta(\beta')$ for some $\beta' < \beta$. Therefore, if $\bar{p}_{<\omega^\delta}(A)(i) = \varphi_\delta(\beta')$, then $\bar{p}_{\omega^\delta}(A)(i) = \beta' < \beta$.

Part (4) is obtained by iterating the third one. □

An important observation about the peeling functions is that each entry of $\bar{p}_\delta(A)$ does not depend on the previous entries of the finite sequence A. In other words $\bar{p}_\delta(A^-) = (\bar{p}_\delta(A))^-$ and $\bar{p}_\delta(A^{-n}) = (\bar{p}_\delta(A))^{-n}$ where A^- denotes the finite sequence A without its first element, and $A^{-(n+1)} = (A^{-n})^-$.

Before defining the coloring promised before Definition 4.5.2, we still need two ingredients. First fix a coloring $d : \alpha \times\!\!\!\times k \to k$ such that for each $i < k$ the preimage $d^{-1}(i)$ has order type α. For each $i < k$ let $\pi_i : d^{-1}(i) \to \alpha$ be the order isomorphism. Second, given a tuple A of ordinals below $\varphi_{\log \gamma}(\alpha \times\!\!\!\times k)$, if $p_\gamma(A) \leqslant p_\gamma(A^-)$, let $\zeta_A \leqslant \gamma$ be the least ordinal ζ such that $p_\zeta(A) \leqslant p_\zeta(A^-)$. As $p_\zeta(A^-)$ is the second entry of $\bar{p}_\zeta(A)$, we have $p_{\zeta_A+1}(A) = 0$. If $p_\gamma(A) > p_\gamma(A^-)$ then ζ_A does not exist and in this case we have $0 < p_\gamma(A) < \alpha \times\!\!\!\times k$ by Lemma 4.5.5.

Definition 4.5.6. Let c be the following $(k+3)$-coloring of the finite tuples of ordinals below $\varphi_{\log \gamma}(\alpha \times\!\!\!\times k)$:

- if ζ_A does not exist, let $c(A) = d(p_\gamma(A)) < k$,
- if ζ_A and ζ_{A^-} both exist and $\zeta_A > \zeta_{A^-}$, let $c(A) = k$,
- if ζ_A and ζ_{A^-} both exist and $\zeta_A = \zeta_{A^-}$, let $c(A) = k+1$,
- If ζ_A exists and either ζ_{A^-} does not or $\zeta_A < \zeta_{A^-}$, let $c(A) = k+2$.

The strategy to prove Theorem 4.5.1 consists in defining a coloring $\bar{c}$ on numbers starting from the coloring c on ordinals and show that homogeneous sets for $\bar{c}$ induce descending sequences in some ordinal, which depends on the color of the homogeneous set. A homogeneous set of color $i < k$ induces a descending sequence in α. A homogeneous set of color k induces a descending sequence in $\gamma + 1$. A homogeneous set of color either $k+1$ or $k+2$ induces a descending sequence of subterms of the first element of the sequence. This imposes a limit on how large the homogeneous sets can be.

Recall that we fixed an ordinal $\mu < \varphi_{\log \gamma}(\alpha \times\!\!\!\times k)$ and that our final goal is to define an infinite set M such that each $s \in [M]^{1 \uplus \mu}$ has a bad coloring on its $(1+\gamma)$-size sets with $k+3$ colors. The set M is a rapidly increasing sequence of natural numbers where the norm function (see Definition 4.1.7) behaves in a particular way. For that we need to define a new norm.

Before defining the new norm, we define a function that assigns to each ordinal $\tau < \varphi_{\log\gamma}(\alpha \divideontimes k)$, a finite set of ordinals $S(\tau)$ that contains all the ordinals which may be equal to ζ_A for some tuple of ordinals A with $A(0) = \tau$.

Definition 4.5.7. We recursively define the set $S(\tau)$ as follows.

- If $\tau = 0$, let $S(\tau) = \{0\}$.
- If $\tau = \varphi_\delta(0)$, let $S(\tau) = \{0, 1, \omega^\delta\}$.
- If $\tau = \varphi_\delta(\sigma)$ with $0 < \sigma < \tau$, let $S(\tau) = \{0,1\} \cup \{\omega^\delta + \xi : \xi \in S(\sigma) \wedge \xi > 0\}$.
- If $\tau = \omega^{\sigma_0} + \ldots + \omega^{\sigma_n}$ with $n > 0$ and $\sigma_0 \geqslant \ldots \geqslant \sigma_n$, let $S(\tau) = \{0,1\} \cup S(\omega^{\sigma_0}) \cup \ldots \cup S(\omega^{\sigma_n})$.

Lemma 4.5.8. *Let $\tau < \varphi_{\log\gamma}(\alpha \divideontimes k)$. Then for each finite tuple of ordinals $A \subset \varphi_{\log\gamma}(\alpha \divideontimes k)$ with $A(0) = \tau$, if ζ_A exists then $\zeta_A \in S(\tau)$.*

Proof. Recall that ζ_A is the least $\zeta \leqslant \gamma$ such that $p_\zeta(A^-) \geqslant p_\zeta(A)$. Because of the minimality of ζ_A, we have that if $\zeta_A > 0$ then $p_{<\zeta_A}(A) > p_{<\zeta_A}(A^-)$, where in the case when ν is a successor, we use $p_{<\nu}(A)$ to mean $p_{\nu-1}(A)$. Since $p_\nu(A^-)$ is non increasing with ν, we must have $p_{<\zeta_A}(A^-) \geqslant p_{\zeta_A}(A^-)$. Putting these three inequalities together, we get

$$p_{<\zeta_A}(A) > p_{\zeta_A}(A).$$

There are two ways by which one could have $p_{<\nu}(A) > p_\nu(A)$ when $\nu > 0$:

(a) a Veblen function was peeled off from $p_{<\nu}(A)$ (possibly φ_0, when ν is successor and we apply $\bar{p}_1$),

(b) ν is successor and $p_{\nu-1}(A) \leqslant p_{\nu-1}(A^-)$.

Notice that the reason for having $p_{<\zeta_A}(A) > p_{\zeta_A}(A)$ cannot be (b), because in this case ζ_A would have been smaller. It follows that ζ_A must be an ordinal at which some peeling was done to $p_{<\zeta_A}(A)$: either an ω was removed (which corresponds to peeling off φ_0) or a φ_δ was removed.

Notice that $0 \in S(\tau)$ for every τ, which takes care of the case when $\tau = A(0) \leqslant A(1)$. So we can assume $A(0) > A(1)$.

Suppose $\tau = \varphi_0(\sigma)$ with $\sigma < \tau$. Then $p_1(A) = \sigma$ and $\zeta_A = 1 + \zeta_B$ where $B = \bar{p}_1(A)$. By inductive hypothesis $\zeta_A \in S(\tau)$.

Suppose now that $\tau = \varphi_\delta(\sigma)$ with $\delta > 0$ and $\sigma < \tau$. Then, for all $\nu < \omega^\delta$, $p_\nu(A) = \tau$ and $p_{\omega^\delta}(A) = \sigma < \tau$. Therefore, either $\zeta_A = \omega^\delta$ or $\zeta_A = \omega^\delta + \zeta_B$ with $\zeta_B > 0$ where $B = \bar{p}_{\omega^\delta}(A)$.

If $\sigma = 0$ then $\zeta_A = \omega^\delta \in S(\varphi_\delta(0))$ and moreover $p_{<\omega^\delta}(A^-) = 0$ must hold otherwise it belongs to ran φ_δ contradicting the minimality of ζ_A. If instead $\sigma > 0$ notice that $p_{\omega^\delta}(A) = \varphi_\delta^{-1}(\tau) > \varphi_\delta^{-1}(p_{<\omega^\delta}(A^-)) = p_{\omega^\delta}(A^-)$ because either $p_{<\omega^\delta}(A^-)$ belongs to the range of φ_δ (which is strictly increasing) or it is 0. It follows that by inductive hypothesis $\zeta_A = \omega^\delta + \zeta_B$ with $\zeta_B \in S(\sigma)$ and $\zeta_B > 0$. Hence $\zeta_A \in S(\tau)$.

Finally, suppose that $A = (\tau, \tau_1, \ldots, \tau_m)$ and that $\tau = \omega^{\sigma_0} + \ldots + \omega^{\sigma_n}$ with $n > 0$. Then $p_1(A)$ is one of the σ_i's, say σ_{i_0}. If $p_1(A) \leqslant p_1(A^-)$, then $\zeta_A = 1 \in S(\tau)$. So suppose that $p_1(A) > p_1(A^-)$ and $\zeta_A \geqslant 2$. The goal now is to define a tuple A' of ordinals such that $A'(0) = \omega^{\sigma_{i_0}}$ and $\zeta_{A'} = \zeta_A$. Then, by the inductive hypothesis we would have $\zeta_{A'} \in S(\omega^{\sigma_{i_0}})$, and so by definition of $S(\tau)$, $\zeta_A \in S(\tau)$. Let $\delta_0 = \sigma_{i_0}$ so that, for some $\delta_1, \ldots, \delta_m$, we have

$$\bar{p}_1(A) = (\overline{\tau, \tau_1}, \overline{\tau_1, \tau_2}, \ldots, \overline{\tau_m, 0}) = (\delta_0, \delta_1, \ldots, \delta_m).$$

Suppose first that $\delta_0 > \delta_1 > \ldots > \delta_m$. Let A' be defined by applying φ_0, which is the exponential function, to all the entries of $\bar{p}_1(A)$. That is, $A' = (\omega^{\delta_0}, \omega^{\delta_1}, \ldots, \omega^{\delta_m})$. Notice that, for all $i < m$,

$$\bar{p}_1(A')(i) = \overline{\omega^{\delta_i}, \omega^{\delta_{i+1}}} = \delta_i = \bar{p}_1(A)(i).$$

Therefore $\bar{p}_1(A') = \bar{p}_1(A)$ and hence $\zeta_{A'} = \zeta_A$. Suppose now that $0 < \ell < m$ is least such that $\delta_\ell \leqslant \delta_{\ell+1}$. We have that

$$\begin{array}{rcccccccc} A & = & (\tau, & \ldots, & \tau_{\ell-1}, & \tau_\ell, & \tau_{\ell+1}, & \ldots, & \tau_m) \\ \bar{p}_1(A) & = & (\delta_0, & \ldots, & \delta_{\ell-1}, & \delta_\ell, & \delta_{\ell+1}, & \ldots, & \delta_m) \\ \bar{p}_2(A) & = & (\overline{\delta_0, \delta_1}, & \ldots, & \overline{\delta_{\ell-1}, \delta_\ell}, & 0, & \overline{\delta_{\ell+1}, \delta_{\ell+2}}, & \ldots, & \overline{\delta_m, 0}) \end{array}$$

Define A' by letting $A'(i) = \omega^{\delta_i}$ for all $i < \ell$, $A'(\ell) = \omega^{\delta_\ell} + \omega^{\delta_\ell}$, $A'(\ell+1) = \omega^{\delta_\ell}$ and $A'(i) = 0$ for all $i > \ell + 1$. We then have that

$$\begin{array}{rccccccccc} A' & = & (\omega^{\delta_0}, & \ldots, & \omega^{\delta_{\ell-1}}, & \omega^{\delta_\ell} + \omega^{\delta_\ell}, & \omega^{\delta_\ell}, & 0, & \ldots, & 0) \\ \bar{p}_1(A') & = & (\delta_0, & \ldots, & \delta_{\ell-1}, & \delta_\ell, & \delta_\ell, & 0, & \ldots, & 0) \\ \bar{p}_2(A') & = & (\overline{\delta_0, \delta_1}, & \ldots, & \overline{\delta_{\ell-1}, \delta_\ell}, & 0, & \overline{\delta_\ell, 0}, & 0, & \ldots, & 0) \end{array}$$

Observe that $\bar{p}_2(A')(\ell) = 0 = \bar{p}_2(A)(\ell)$ and hence that $\bar{p}_\nu(A')(\ell) = 0 = \bar{p}_\nu(A)(\ell)$ for all $\nu \geqslant 2$. Observe also that $\bar{p}_2(A')(i) = \bar{p}_2(A)(i)$ for all $i \leqslant \ell$. It is not hard to see that each application of $\bar{p}_1$ keeps equal the two sequences up to the ℓ-th entry while each application of $\bar{p}_{\omega^\nu}$ for some ν just peels off (when possible) an application of φ_ν from each entry independently on the other entries of the sequence. Therefore we get that $\bar{p}_\nu(A')(i) = \bar{p}_\nu(A)(i)$ for all $i \leqslant \ell$ and all $\nu \geqslant 2$. It follows that $\zeta_{A'} = \zeta_A$ as needed. □

Remark 4.5.9. Notice that if $\zeta_A = \omega^{\delta_0} + \ldots + \omega^{\delta_n}$ is a limit ordinal, then $p_{<\zeta_A}(A) = \varphi_{\delta_n}(0)$ and $p_{<\zeta_A}(A^-) = 0$. To see this, let $B = \bar{p}_{<\zeta_A}(A) = \bar{p}_{<\omega^{\delta_n}} \circ \bar{p}_{\omega^{\delta_{n-1}}} \circ \ldots \circ \bar{p}_{\omega^{\delta_0}}(A)$ so that $\bar{p}_{\zeta_A}(A) = \varphi_{\delta_n}^{-1}(B)$. By definition of ζ_A it must be $B(0) > B(1)$ and by definition of $\bar{p}_{<\omega^{\delta_n}}$ both $B(0)$ and $B(1)$ must be either 0 or in the range of φ_{δ_n}. If $B(1) > 0$ then since φ_{δ_n} is strictly increasing it must be $\varphi_{\delta_n}^{-1}(B(0)) > \varphi_{\delta_n}^{-1}(B(1))$, contradicting the definition of ζ_A. Then $B(1) = 0$ and $\varphi_{\delta_n}^{-1}(B(0)) \leqslant \varphi_{\delta_n}^{-1}(B(1)) = 0$ which implies $B(0) = \varphi_{\delta_n}(0)$.

We defined the norm $|\delta|$ of an ordinal δ in Definition 4.1.7. We now use that norm to define a new function $||\cdot|| : \varphi_{\log\gamma}(\alpha \divideontimes k) \to \omega$ as follows.

Definition 4.5.10. For each ordinal $\beta < \varphi_{\log\gamma}(\alpha \divideontimes k)$, let $||\beta||$ be 1 plus the maximum of the following:

- the norm of all the subterms of β, including β itself,
- the cardinality of the multiset $\mathrm{Sub}(\beta)$,
- the norm of all the ordinals in $S(\beta)$,
- for every $\delta \in \mathrm{Sub}(\beta)$ such that $\delta < \alpha \divideontimes k$, the norm of $\pi_{d(\delta)}(\delta)$.

We now show that we just need to know the first entry of a finite sequence of ordinals A to compute a level such that $p_{<\omega^\sigma}(A)$ stabilizes.

Lemma 4.5.11. *Let $\sigma > 0$ be an ordinal and let $A = (\alpha_0, \ldots, \alpha_m)$ be a sequence of ordinals in $\varphi_{\log\gamma}(\alpha \divideontimes k)$. Then,*

$$p_{<\omega^\sigma}(A) = p_{\omega^{\sigma[||\alpha_0||]} \cdot ||\alpha_0||}(A).$$

Proof. First we claim that for some $\xi < \omega^{\sigma[||\alpha_0||]} \cdot ||\alpha_0||$ we have that $p_\xi(A)$ is either 0 or a fixed point of $\varphi_{\sigma[||\alpha_0||]}$. To prove this, notice that by Lemma 4.5.5 we know that each application of $p_{\omega^{\sigma[||\alpha_0||]}}$ peels off at least one application of $\varphi_{\sigma[||\alpha_0||]}$ to each entry of its input, unless we have already got to 0 or to a fixed point of $\varphi_{\sigma[||\alpha_0||]}$. The function $p_{\omega^{\sigma[||\alpha_0||]} \cdot ||\alpha_0||}$ peels off at least $||\alpha_0||$ applications of $\varphi_{\sigma[||\alpha_0||]}$. However, by definition of $||\alpha_0||$ (which also counts the cardinality of $\mathrm{Sub}(\alpha_0)$) the number of operations $\varphi_{\sigma[||\alpha_0||]}$ occurring in α_0 is strictly less than $||\alpha_0||$. For this reason $p_\xi(A)$ must be 0 or a fixed point of $\varphi_{\sigma[||\alpha_0||]}$ for some $\xi < \omega^{\sigma[||\alpha_0||]} \cdot ||\alpha_0||$ as claimed.

If $p_\xi(A) = 0$ then the sequence has already stabilized. Otherwise $p_\xi(A)$ is a fixed point of $\varphi_{\sigma[||\alpha_0||]}$ and can be uniquely written as $\varphi_\delta(\beta)$ for some $\delta > \sigma[||\alpha_0||]$ and some $\beta < \varphi_\delta(\beta)$. Since $p_\xi(A)$ is a subterm of α_0 (by Lemma 4.5.5) we get that $|\delta| \leqslant ||\alpha_0||$. Then, if $\delta < \sigma$, the goodness of the norm $|\cdot|$ implies that $\sigma \Rightarrow_{||\alpha_0||} \delta$ by Lemma 4.1.11. It follows that $\sigma[||\alpha_0||] \geqslant \delta$ which is a contradiction. We conclude that $\delta \geqslant \sigma$, and hence $p_\xi(A) \in \mathrm{ran}(\varphi_\sigma)$.

If $p_\xi(A) \leqslant p_\xi(A^-)$ then $p_{\xi+1}(A) = 0$ and hence the sequence stabilizes at $\xi+1$. If instead $p_\xi(A) > p_\xi(A^-)$ then by induction we can prove that $p_\rho(A) = p_\xi(A) > p_\rho(A^-)$ for any $\xi \leqslant \rho < \omega^\sigma$: this is because $p_\rho(A^-) \leqslant p_\xi(A^-)$ and at stage ρ we attempt to peel off $\varphi_{\sigma'}$ for some $\sigma' < \sigma$, which has no effect on $p_\xi(A)$.

Since $\xi + 1 \leqslant \omega^{\sigma[||\alpha_0||]} \cdot ||\alpha_0||$, we have $p_{<\omega^\sigma}(A) = p_{\omega^{\sigma[||\alpha_0||]} \cdot ||\alpha_0||}(A)$. □

We are ready to construct the set M of Theorem 4.5.1.

Lemma 4.5.12. *There exists an infinite set* M *such that, for all* $s \in [M]^{<\omega}$*, we have that for all* $i < |s|$

$$\max(||\mu[s \restriction i]||, ||\omega||, ||\gamma+1||) + 2 < s(i).$$

Proof. We define $M(n)$ by recursion on n. Let $M(0)$ be any number which is larger than $\max(||\mu||, ||\omega||, ||\gamma+1||) + 2$. For each $n > 0$ let $M(n) > M(n-1)$ be a number strictly larger than $||\mu[s]|| + 2$ for all $s \subseteq \{M(0), \ldots, M(n-1)\}$.

Checking that M is the required set is straightforward: if $s \in [M]^{<\omega}$ and $i < |s|$, then $s(i) = M(n)$ for some n and $M(n) > ||\mu[s \restriction i]|| + 2$ by definition of $M(n)$. Moreover, $s(i)$ is greater than $||\omega|| + 2$ and $||\gamma+1|| + 2$ because $M(0)$ is. □

For the rest of this section fix a set $s \in [M]^{1 \uplus \mu}$, where M is as in the previous lemma. Define

$$T\colon s \to \varphi_{\log \gamma}(\alpha \times\!\!\!\times k) \quad \text{by} \quad T(s(i)) = \mu[s \restriction i].$$

Notice that $T(\min s) = \mu$, $T(\max s) = 0$ (as s^* is μ-size) and that T is a strictly decreasing function. Moreover, by definition of M,

$$||T(s(i))|| + 2 < s(i) \qquad \text{for all } i < |s|.$$

Now we have all the required machinery. We define a coloring on $[s]^{\geqslant 1+\gamma}$ (which denotes the set of $(1+\gamma)$-large subsets of s)

$$\begin{aligned} \bar{c}\colon\ [s]^{\geqslant 1+\gamma} &\to k+3 \\ u &\mapsto c(T(u)) \end{aligned}$$

where c is the coloring from Definition 4.5.6. By $T(u_0, \ldots, u_m)$ we mean $(T(u_0), \ldots, T(u_m))$. Similarly, we define $\bar{p}_\delta$ and ζ on $u \subseteq s$ by $\bar{p}_\delta(u) = \bar{p}_\delta(T(u))$ and $\zeta_u = \zeta_{T(u)}$.

We now show that in the definition of $\bar{c}$, only the $(1+\gamma)$-size prefix matters: i.e. if $t \subseteq s$ and $v \sqsubseteq t$ is the $(1+\gamma)$-size initial segment of t, then $\bar{c}(t) = \bar{c}(v)$.

Lemma 4.5.13. *For each* ν *and sets* $u \sqsubseteq t \subseteq s$*, if* u *is* $(1+\nu)$*-large then*

$$p_\nu(u) = p_\nu(t).$$

When $\nu = \gamma$ this shows that the coloring $\bar{c}$ defined above can actually be regarded as a coloring of the $(1+\gamma)$-size subsets of s. This is because in the coloring $\bar{c}$ of a set u we only care about the peeling functions of index γ or index $\nu \in S(T(\min u))$. By definition of the set M (which contains u) each element of u is larger than $\|T(\min u)\|$ and by definition of $\|\cdot\|$, it is also larger than $|\nu|$ for each $\nu \in S(T(\min u))$. This means that if u is γ-large, since $\gamma \geqslant \nu$, u must also be ν-large and so Lemma 4.5.13 yields that also p_ν only depends on the $(1+\nu)$-size initial segment of its input.

Proof. We proceed by induction on ν. The case $\nu = 0$ is immediate since $\bar{p}_0$ is the identity function. If $\nu = \rho + 1$ and u is $(1+\nu)$-large then u is $(1+\rho+1)$-large and u^- is $(1+\rho)$-large. By Corollary 4.1.4 u is $(1+\rho)$-large too. Hence we have both $p_\rho(u) = p_\rho(t)$ and $p_\rho(u^-) = p_\rho(t^-)$ by inductive hypothesis. Then

$$p_\nu(u) = \overline{p_\rho(u), p_\rho(u^-)} = \overline{p_\rho(t), p_\rho(t^-)} = p_\nu(t).$$

Suppose now $\nu = \omega^\sigma > 1$. Recall that we denote $\min u$ by u_0. Suppose that u_0 is the i-th element of s. First notice that the first ordinal of the sequence to which we are applying $\bar{p}_\nu$ is $T(u_0) = \mu[s \restriction i]$ and so by Lemma 4.5.12 we have that $u_0 = s(i) > \|T(u_0)\| + 2$. Since u is $(1+\omega^\sigma)$-large then it is $(1+\omega^\sigma[u_0])$-large too (this is because it contains u^- which is $(1+\omega^\sigma[u_0])$-large). We claim that $\omega^\sigma[u_0] > \omega^{\sigma[\|T(u_0)\|]} \cdot \|T(u_0)\|$. We proceed by cases.

(1) If $\sigma < \omega^\sigma$ then

$$\omega^\sigma[u_0] = \omega^{\sigma[u_0]} \cdot u_0 > \omega^{\sigma[\|T(u_0)\|]} \cdot \|T(u_0)\|.$$

(2) If $\sigma = \varphi_1(0)$ then

$$\begin{aligned}\omega^\sigma[u_0] = \varphi_1(0)[u_0] = \varphi_0^{u_0+1}(0) &> \varphi_0^{\|T(u_0)\|+3}(0)\\ &> \varphi_0^{\|T(u_0)\|+2}(0) \cdot \|T(u_0)\| = \omega^{\sigma[\|T(u_0)\|]} \cdot \|T(u_0)\|.\end{aligned}$$

(3) If $\sigma = \varphi_1(\beta) > \beta > 0$ then

$$\begin{aligned}\omega^\sigma[u_0] = \varphi_1(\beta)[u_0] &= \varphi_0^{u_0+1}(\varphi_1(\beta[u_0]) + 1)\\ &> \varphi_0^{\|T(u_0)\|+3}(\varphi_1(\beta[\|T(u_0)\| + 2]) + 1)\\ &> \varphi_0^{\|T(u_0)\|+2}(\varphi_1(\beta[\|T(u_0)\|]) + 1) \cdot \|T(u_0)\|\\ &= \omega^{\sigma[\|T(u_0)\|]} \cdot \|T(u_0)\|.\end{aligned}$$

(4) If $\sigma = \varphi_\delta(0)$ with $\delta > 1$ then

$$\omega^\sigma[u_0] = \varphi_\delta(0)[u_0] = \varphi_{\delta[u_0]}^{u_0+1}(0) > \varphi_{\delta[||T(u_0)||+2]}^{||T(u_0)||+3}(0)$$
$$> \varphi_{\delta[||T(u_0)||]}^{||T(u_0)||+1}(0) \cdot ||T(u_0)|| = \omega^{\sigma[||T(u_0)||]} \cdot ||T(u_0)||.$$

(5) If $\sigma = \varphi_\delta(\beta) > \beta > 0$ with $\delta > 1$ then

$$\begin{aligned}\omega^\sigma[u_0] &= \varphi_\delta(\beta)[u_0] = \varphi_{\delta[u_0]}^{u_0+1}(\varphi_\delta(\beta[u_0]) + 1)\\ &> \varphi_{\delta[||T(u_0)||+2]}^{||T(u_0)||+3}(\varphi_1(\beta[||T(u_0)|| + 2]) + 1)\\ &> \varphi_{\delta[||T(u_0)||]}^{||T(u_0)||+1}(\varphi_1(\beta[||T(u_0)||]) + 1) \cdot ||T(u_0)||\\ &= \omega^{\sigma[||T(u_0)||]} \cdot ||T(u_0)||.\end{aligned}$$

(6) If $\sigma = \Gamma_0$ then

$$\begin{aligned}\omega^\sigma[u_0] &= \Gamma_0[u_0] = \varphi_{\Gamma_0[u_0-1]}(0) > \varphi_{\Gamma_0[||T(u_0)||+1]}(0)\\ &> \varphi_{\Gamma_0[||T(u_0)||]}(0) \cdot ||T(u_0)|| = \Gamma_0[||T(u_0)|| + 1] \cdot ||T(u_0)||\\ &= \omega^{\sigma[||T(u_0)||]} \cdot ||T(u_0)||.\end{aligned}$$

(7) If $\sigma = \Gamma_\xi$ with $0 < \xi < \zeta$ then

$$\begin{aligned}\omega^\sigma[u_0] &= \Gamma_\xi[u_0]\\ &= \varphi_{\cdot^{\cdot^{\cdot \varphi_{\Gamma_{\xi[u_0]}+1}(0) \cdot^{\cdot^{\cdot}}}}}(0) && u_0 + 1 \text{ times}\\ &> \varphi_{\cdot^{\cdot^{\cdot \varphi_{\Gamma_{\xi[||T(u_0)||]}+1}(0) \cdot^{\cdot^{\cdot}}}}}(0) && ||T(u_0)|| + 2 \text{ times}\\ &> \varphi_{\cdot^{\cdot^{\cdot \varphi_{\Gamma_{\xi[||T(u_0)||]}+1}(0) \cdot^{\cdot^{\cdot}}}}}(0) \cdot ||T(u_0)|| && ||T(u_0)|| + 1 \text{ times}\\ &= \Gamma_\xi[||T(u_0)||] \cdot ||T(u_0)||\\ &= \omega^{\sigma[||T(u_0)||]} \cdot ||T(u_0)||.\end{aligned}$$

We know by Lemma 4.5.11 that $p_{<\omega^\sigma}(u) = p_{\omega^{\sigma[||T(u_0)||]} \cdot ||T(u_0)||}(u)$ and also that $p_{<\omega^\sigma}(t) = p_{\omega^{\sigma[||T(u_0)||]} \cdot ||T(u_0)||}(t)$. By the claim and by Lemma 4.5.11, we know $p_{\omega^{\sigma[||T(u_0)||]} \cdot ||T(u_0)||}(u) = p_{\omega^\sigma[u_0]}(u)$ and $p_{\omega^{\sigma[||T(u_0)||]} \cdot ||T(u_0)||}(t) = p_{\omega^\sigma[u_0]}(t)$. Finally, since u is $(1 + \omega^\sigma[u_0])$-large, by inductive hypothesis we also know that $p_{\omega^\sigma[u_0]}(u) = p_{\omega^\sigma[u_0]}(t)$ and so we get $p_{<\omega^\sigma}(u) = p_{<\omega^\sigma}(t)$. We then peel off one application of φ_σ in each side and obtain $p_{\omega^\sigma}(u) = p_{\omega^\sigma}(t)$ as required.

We are left with case $\nu = \rho + \omega^\sigma$ with $\rho \ggg \omega^\sigma$ and $\sigma > 0$. Recall $p_{<\omega^\sigma}(\bar{p}_\rho(u)) = \lim_{\tau \to \omega^\sigma} p_\tau(\bar{p}_\rho(u))$ and by Lemma 4.5.11 the limit converges after the ordinal $\omega^{\sigma[||p_\rho(u)||]}$.

$\|p_\rho(u)\|$. Analogously $p_{<\omega^\sigma}(\bar{p}_\rho(t)) = \lim_{\tau\to\omega^\sigma} p_\tau(\bar{p}_\rho(t))$ and it converges after $\omega^{\sigma[\|p_\rho(t)\|]} \cdot \|p_\rho(t)\|$. Since $p_\rho(u)$ and $p_\rho(t)$ are subterms of $T(u_0)$ (by Lemma 4.5.5) then $\|T(u_0)\| \geqslant \|p_\rho(u)\|$ and $\|T(u_0)\| \geqslant \|p_\rho(t)\|$. Therefore, by the claim within the proof of case $\nu = \omega^\sigma$ above, we get that $\omega^\sigma[u_0] > \omega^{\sigma[\|p_\rho(u)\|]} \cdot \|p_\rho(u)\|$ and $\omega^\sigma[u_0] > \omega^{\sigma[\|p_\rho(t)\|]} \cdot \|p_\rho(t)\|$. Now u is $(1+\rho+\omega^\sigma[u_0])$-large because u^- is. Hence, by inductive hypothesis, $p_{\rho+\omega^\sigma[u_0]}(u) = p_{\rho+\omega^\sigma[u_0]}(t)$. We get that $p_{<\omega^\sigma}(\bar{p}_\rho(u)) = p_{<\omega^\sigma}(\bar{p}_\rho(t))$ and peeling off one application of φ_σ in each side we obtain $p_{\rho+\omega^\sigma}(u) = p_{\omega^\sigma}(\bar{p}_\rho(u)) = p_{\omega^\sigma}(\bar{p}_\rho(t)) = p_{\rho+\omega^\sigma}(t)$ as required. □

We can now go back to the proof of Theorem 4.5.1.

Proof of Theorem 4.5.1. Towards a contradiction suppose that t is a $((1+\gamma)\uplus\alpha)$-large homogeneous subset of s for the coloring $\bar{c}$ we defined above. Write t as $t_\alpha{}^\frown t_\gamma$ with $t_\alpha < t_\gamma$, where t_α is α-size and t_γ is $(1+\gamma)$-large. For each $i \leqslant |t_\alpha|$, let t^{-i} be the set obtained by removing the first i elements from t. Since $t_\gamma \subseteq t^{-i}$, we have that t^{-i} is $(1+\gamma)$-large. By homogeneity, we have that $\bar{c}(t^{-i})$ has the same color for all $i \leqslant |t_\alpha|$. We distinguish four cases based on the color of the homogeneous set t.

Case 1: Suppose that t is homogeneous of color $j < k$. Then, for every $i \leqslant |t_\alpha|$, we have that $p_\gamma(t^{-i}) \in d^{-1}(j)$. Recall that $d^{-1}(j)$ is a subset of $\alpha \divideontimes k$ isomorphic to α and that we denoted by $\pi_j : d^{-1}(j) \to \alpha$ the order isomorphism. Define a map

$$\begin{aligned} f\colon\ & t_\alpha \cup \{t_\gamma(0)\} \to \alpha \\ & t(i) \mapsto \pi_j(p_\gamma(t^{-i})). \end{aligned}$$

Recall that since $\bar{c}(t^{-i}) = j < k$, we have that $\zeta_{t^{-i}}$ does not exists, and hence that $p_\gamma(t^{-i}) > p_\gamma(t^{-(i+1)})$. Therefore f is strictly decreasing. Furthermore, the ordinal $p_\gamma(t^{-i})$ is a subterm of $T(t(i))$ which belongs to $d^{-1}(j)$ and hence $|\pi_j(p_\gamma(t^{-i}))| < \|T(t(i))\| < t(i)$ by Definition 4.5.10 and by definition of the set M in Lemma 4.5.12. In other words, for all $m \in t_\alpha \cup \{t_\gamma(0)\}$, $|f(m)| < m$. The Estimation Lemma 4.1.12 states that no such f can exist because $t_\alpha \cup \{t_\gamma(0)\}$ is $(1 \uplus \alpha)$-size.

Case 2: Suppose that t is homogeneous of color k. In this case we have that

$$\zeta_{t^{-0}} > \zeta_{t^{-1}} > \cdots > \zeta_{t^{-|t_\alpha|}}$$

and all these ordinals belong to $\gamma+1$. We now define a map

$$\begin{aligned} f\colon\ & t_\alpha \cup \{t_\gamma(0)\} \to \gamma+1 \\ & t(i) \mapsto \zeta_{t^{-i}}. \end{aligned}$$

To apply the Estimation Lemma, we need to show that $t_\alpha \cup \{t_\gamma(0)\}$ is $(1 \uplus (\gamma+1))$-large which is equivalent to t_α being $(\gamma+1)$-large. First we notice that it is $(1 \uplus \alpha)$-large since t_α is α-size. Then recall that we are assuming $\alpha > \gamma$ which is equivalent to $\alpha \geqslant \gamma+1$. If $\alpha = \gamma+1$ we are done. If $\alpha > \gamma+1$ then by Lemma 4.1.9 and because $t \subset M$ (and $M(0) > |\gamma+1|$ by Lemma 4.5.12) we know that for each i if $\alpha[t(0), \dots, t(i)] > \gamma+1$ then $\alpha[t(0), \dots, t(i+1)] \geqslant \gamma+1$. Since $\alpha[t_\alpha] = 0$ there exists $i < |t_\alpha|$ such that $\alpha[t(0), \dots, t(i)] = \gamma+1$. Therefore t_α is $(\gamma+1)$-large.

Notice that, by Lemma 4.5.8, $\zeta_{t^{-i}} \in S(T(t(i)))$. It follows that for $m = t(i)$,

$$|f(m)| = |\zeta_{t^{-i}}| \leqslant \|T(t(i))\| < t(i) = m.$$

The Estimation Lemma states that no such f exists.

Case 3: Suppose that t is homogeneous of color $k+1$. In this case we have

$$\zeta_{t^{-0}} = \zeta_{t^{-1}} = \dots = \zeta_{t^{-|t_\alpha|}}$$

and let ζ denote this ordinal. Notice that $\zeta \neq 0$ because $T(t)$ is a strictly decreasing sequence of ordinals.

We claim that the ordinals $p_\zeta(t^{-i})$ originate from distinct occurrences of the elements of the multiset $\mathrm{Sub}(T(t(0)))$ and hence there should be no more than $\|T(t(0))\|$ many of them. Since $\|T(t(0))\| < t(0)$ by Lemma 4.5.12, this implies that t_α has less than $t(0)$ elements. However, since $\alpha \geqslant \omega$ and $M(0) > |\omega|$, we get that t_α is ω-large (the argument is the same used to show that t_α is $(\gamma+1)$-large in Case 2 above), which contradicts the fact that t_α has less than $t(0)$ elements.

Let us now prove the claim. We have

$$\begin{aligned} & p_{<\zeta}(t^{-0}) > p_{<\zeta}(t^{-1}) > \dots > p_{<\zeta}(t^{-|t_\alpha|}) > p_{<\zeta}(t^{-|t_\alpha|-1}) \\ \text{and} \quad & p_\zeta(t^{-0}) \leqslant p_\zeta(t^{-1}) \leqslant \dots \leqslant p_\zeta(t^{-|t_\alpha|}) \leqslant p_\zeta(t^{-|t_\alpha|-1}). \end{aligned}$$

First we show that ζ must be a successor ordinal so that $p_{<\zeta}$ is $p_{\zeta-1}$. Towards a contradiction suppose that ζ is limit. By Remark 4.5.9, it must be $p_{<\zeta}(t^{-0}) = \varphi_\delta(0)$ for some $\delta > 0$ and $p_{<\zeta}(t^{-1}) = 0$. In particular, the strictly decreasing chain above must have length 2 and so $t_\alpha = \emptyset$, a contradiction.

Going back to the proof of the claim, recall that, for every $i \leqslant |t_\alpha|$, $p_\zeta(t^{-i})$ is an exponent of a term in the Cantor normal form of $p_{\zeta-1}(t^{-i})$. We claim that for this to be the case, we must have the following situation:

$$
\begin{aligned}
p_{\zeta-1}(t^{-0}) &= \omega^{\alpha_0} + \cdots + \omega^{\alpha_{n_1}} + \omega^{p_\zeta(t^{-1})} + \cdots + \omega^{\alpha_{n_0}} + \omega^{p_\zeta(t^{-0})} + \tau_0 \\
p_{\zeta-1}(t^{-1}) &= \omega^{\alpha_0} + \cdots + \omega^{\alpha_{n_1}} + \omega^{p_\zeta(t^{-1})} + \cdots + \omega^{\alpha_{n_0}} + \tau_1 \\
p_{\zeta-1}(t^{-2}) &= \omega^{\alpha_0} + \cdots + \omega^{\alpha_{n_1}} + \tau_2 \\
\vdots \quad & \quad \vdots \quad \quad \vdots
\end{aligned}
$$

By definition of $p_\zeta(t^{-0})$, we know that the Cantor normal forms of the ordinals $p_{\zeta-1}(t^{-0})$ and $p_{\zeta-1}(t^{-1})$ are equal up to a certain term $\omega^{\alpha_{n_0}}$, and then in $p_{\zeta-1}(t^{-0})$ we have the term $\omega^{p_\zeta(t^{-0})}$, while the tail τ_1 of $p_{\zeta-1}(t^{-1})$ is strictly smaller than $\omega^{p_\zeta(t^{-0})}$. Since $p_\zeta(t^{-0}) \leqslant p_\zeta(t^{-1})$, we must have that $p_\zeta(t^{-1})$ is an exponent of the Cantor normal form of $p_{\zeta-1}(t^{-1})$ that shows up before n_0. This implies that $p_\zeta(t^{-1})$ is one of the exponents of $p_{\zeta-1}(t^{-0})$ too. Since $\omega^{p_\zeta(t^{-0})} > \tau_1$, we must have that $\omega^{p_\zeta(t^{-1})}$ comes strictly before $\omega^{p_\zeta(t^{-0})}$ in the Cantor normal form of $p_{\zeta-1}(t^{-0})$. That is, even if they are equal as ordinals, they are different subterms of $p_{\zeta-1}(t^{-0})$. Continuing like this we get that each $p_\zeta(t^{-i})$ is a different exponent in the Cantor normal form of $p_{\zeta-1}(t^{-0})$. Finally, recall that $p_{\zeta-1}(t^{-0})$ is already a subterm of $T(t(0))$ by Lemma 4.5.5. Therefore, we get that that each $p_\zeta(t^{-i})$ for $i \leqslant |t_\alpha|$ originates from a distinct occurrence of an element of $\mathrm{Sub}(T(t(0)))$, as required.

Case 4: Suppose that t is homogeneous of color $k+2$. In this case we have

$$\zeta_{t^{-0}} < \zeta_{t^{-1}} < \zeta_{t^{-2}} < \cdots < \zeta_{t^{-|t_\alpha|}}.$$

Notice that, by definition of the color $k+2$, they must all be defined while $\zeta_{t^{-|t_\alpha|-1}}$ might be undefined. For each i, let us denote $\zeta_{t^{-i}}$ by ζ_i for simplicity. Again we first show that each ζ_i is a successor ordinal. Each of them must be non zero because $T(t)$ is strictly decreasing. Towards a contradiction suppose that ζ_i is limit. Then by Remark 4.5.9, it must be $p_{<\zeta_i}(t^{-i}) = \varphi_\delta(0)$ for some $\delta > 0$ and $p_{<\zeta_i}(t^{-i-1}) = 0$. But in this case ζ_{i+1} must be defined and strictly smaller than ζ_i, a contradiction.

We claim that for each $i > 0$, $p_{\zeta_i-1}(t^{-i})$ is a proper subterm of $p_{\zeta_{i-1}-1}(t^{-i+1})$. It would then again follow that the ordinals $p_{\zeta_i}(t^{-i})$ are all different subterms of $T(t(0))$ and we would get the same contradiction as in the previous case. For ease the notation we only consider the case $i = 1$. Since $p_{\zeta_0-1}(t^{-0}) > p_{\zeta_0-1}(t^{-1})$ and $p_{\zeta_0}(t^{-0}) \leqslant p_{\zeta_0}(t^{-1})$, as in Case 3, we have that $p_{\zeta_0}(t^{-1})$ is an exponent of the Cantor normal form of $p_{\zeta_0-1}(t^{-0})$ that shows up before $p_{\zeta_0}(t^{-0})$. Since $\zeta_1 > \zeta_0$, we have that $p_{\zeta_1-1}(t^{-1})$ is a subterm of $p_{\zeta_0}(t^{-1})$. We thus have that $p_{\zeta_1-1}(t^{-1})$ is a proper subterm of $p_{\zeta_0-1}(t^{-0})$. □

4.6 THE UPPER BOUND

In this section we show that for each $k \in \mathbb{N}$ and every countable ordinals α and γ, $\mathrm{Ram}(\alpha)^{1+\gamma}_{<\omega} \leqslant \varphi_{\log\gamma}(\alpha \cdot \omega)$. This is achieved by proving the following theorem, which is the main result of this section.

Theorem 4.6.1. *Let $k \in \mathbb{N}$ and α and γ be countable ordinals. Then*

$$\mathrm{Ram}(\alpha)^{1+\gamma}_{k} \leqslant \varphi_{\log\gamma}(\alpha \divideontimes k).$$

Recall that this means that for all SD-barriers $\mathcal{A}$ and $\mathcal{C}$ with $\mathrm{ht}(\mathcal{A}) = \alpha$, $\mathrm{ht}(\mathcal{C}) = 1+\gamma$ and $\mathrm{base}(\mathcal{A})$ a final segment of $\mathrm{base}(\mathcal{C})$, there exists a SD-barrier $\mathcal{B}$ such that $\mathrm{base}(\mathcal{B}) = \mathrm{base}(\mathcal{A})$, $\mathrm{ht}(\mathcal{B}) \leqslant \varphi_{\log\gamma}(\alpha \divideontimes k)$ and $\mathcal{B} \to (\mathcal{A})^{\mathcal{C}}_{k}$.

Since $\sup_{k\in\mathbb{N}} \alpha \divideontimes k = \alpha \cdot \omega$ and the Veblen functions are normal, Theorem 4.6.1 implies that $\mathrm{Ram}(\alpha)^{1+\gamma}_{<\omega} \leqslant \varphi_{\log\gamma}(\alpha \cdot \omega)$.

Notice that if $\gamma = 0$, by Corollary 4.4.6 we already know that $\mathrm{Ram}(\alpha)^{1}_{k} = \alpha \divideontimes k$. Since by Definition 4.3.7 $\varphi_{\log 0}$ is the identity function we obtain Theorem 4.6.1 in this case and from now on we can assume that $\gamma \geqslant 1$. Moreover, if both α and γ are below ω then $\mathrm{Ram}(\alpha)^{1+\gamma}_{k} < \omega \leqslant \varphi_{\log\gamma}(\alpha \divideontimes k)$. Hence we can restrict to the case $\max\{\alpha,\gamma\} \geqslant \omega$.

First we show that we can reduce to a simpler case, useful for the rest of the section.

Lemma 4.6.2. *Let $k \in \mathbb{N}$ and $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be fronts where $\mathrm{base}(\mathcal{B}) = \mathrm{base}(\mathcal{A})$ is a final segment of $\mathrm{base}(\mathcal{C})$. If $\mathcal{B} \to (\mathcal{A})^{\mathbb{1}\oplus\mathcal{C}}_{k}$ then $\mathcal{B} \to (\mathcal{A})^{\mathcal{C}}_{k}$*

Proof. Let s be a $(\mathbb{1}\oplus\mathcal{B})$-size set and let $c\colon [s]^{\mathcal{C}} \to k$ be a coloring. Define the coloring $c'\colon [s]^{\mathbb{1}\oplus\mathcal{C}} \to k$ as $u \mapsto c(u^*)$. Then by assumption there exists a set $t \in [s]^{\mathbb{1}\oplus\mathcal{C}\oplus\mathcal{A}}$ c'-homogeneous, say of color $i < k$. By definition t^* is $(\mathcal{C}\oplus\mathcal{A})$-size: we claim that t^* is c-homogeneous of color i. Let $u \in [t^*]^{\mathcal{C}}$. Then $\max t > \max u$ and so $u^\frown\langle\max t\rangle \in [t]^{\mathbb{1}\oplus\mathcal{C}}$. It follows that $c(u) = c'(u^\frown\langle\max t\rangle) = i$. ☐

We need the following extension of the arrow notation.

Definition 4.6.3. Let $k \in \mathbb{N}$ and $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ be fronts such that $\mathrm{base}(\mathcal{B}) = \mathrm{base}(\mathcal{A})$ and $\mathrm{base}(\mathcal{D})$ are final segments of $\mathrm{base}(\mathcal{C})$. We write

$$\mathcal{B} \to (\mathcal{A})^{\mathbb{1}\oplus\mathcal{C}\oplus\mathcal{D}\to\mathbb{1}\oplus\mathcal{D}}_{k}$$

to mean that for each $(\mathbb{1}\oplus\mathcal{B})$-size s and each k-coloring c of $[s]^{\mathbb{1}\oplus\mathcal{C}\oplus\mathcal{D}}$, there exists a $(\mathbb{1}\oplus\mathcal{C}\oplus\mathcal{A})$-size t subset of s such that the coloring of an element of $[t]^{\mathbb{1}\oplus\mathcal{C}\oplus\mathcal{D}}$ depends only on its $(\mathbb{1}\oplus\mathcal{D})$-size initial segment. We say that t is a $(\mathbb{1}\oplus\mathcal{D})$-prehomogeneous set for c.

Next we state a key intermediate result to prove Theorem 4.6.1.

Theorem 4.6.4. *Let* $k \in \mathbb{N}$ *and* $\mathcal{A}$ *and* $\mathcal{C}$ *be* SD*-barriers where* $\mathrm{ht}(\mathcal{A}) = \alpha$, $\mathrm{ht}(\mathcal{C}) = \gamma$ *and* $\mathrm{base}(\mathcal{A})$ *is a final segment of* $\mathrm{base}(\mathcal{C})$. *There exists a* SD*-barrier* $\mathcal{B}$ *with* $\mathrm{base}(\mathcal{B}) = \mathrm{base}(\mathcal{A})$ *and* $\mathrm{ht}(\mathcal{B}) \leqslant \varphi_{\log\gamma}(\alpha)$ *such that* $\mathcal{B} \to (\mathcal{A})_k^{\mathbb{1}\oplus\mathcal{C}\to\mathbb{1}}$.

We delay the proof of Theorem 4.6.4 to the end of the section. We show first how to derive Theorem 4.6.1.

Proof of Theorem 4.6.1. Recall that $\mathcal{A}$ and $\mathcal{C}$ are SD-barriers with $\mathrm{ht}(\mathcal{A}) = \alpha$, $\mathrm{ht}(\mathcal{C}) = 1+\gamma$ and $\mathrm{base}(\mathcal{A})$ a final segment of $\mathrm{base}(\mathcal{C})$. Recall also that we assume $\gamma \geqslant 1$. If γ is finite then by decomposability $\mathcal{C} = [\mathrm{base}(\mathcal{C})]^{1+\gamma}$. Therefore $\mathcal{C} = \mathbb{1} \oplus \mathcal{C}'$ where $\mathcal{C}' = [\mathrm{base}(\mathcal{C})]^{\gamma}$. If γ is infinite then $\mathrm{ht}(\mathcal{C}) = 1+\gamma = \gamma$ and in this case let $\mathcal{C}' = \mathcal{C}$.

Let $\mathcal{B}(\mathcal{A}) = \mathcal{B}(\mathcal{A}, \dots, \mathcal{A})$ be the SD-barrier of Definition 4.4.2 associated to k many $\mathcal{A}$'s. By Lemma 4.4.4 $\mathrm{ht}(\mathcal{B}(\mathcal{A})) = \alpha \otimes k$ and by Theorem 4.4.5 $\mathcal{B}(\mathcal{A}) \to (\mathcal{A})_k^{\mathbb{1}}$. By Theorem 4.6.4 there exists a SD-barrier $\mathcal{B}$ such that $\mathrm{ht}(\mathcal{B}) \leqslant \varphi_{\log\gamma}(\alpha \otimes k)$ and $\mathcal{B} \to (\mathcal{B}(\mathcal{A}))_k^{\mathbb{1}\oplus\mathcal{C}'\to\mathbb{1}}$. We claim that it suffices to prove that $\mathcal{B} \to (\mathcal{A})_k^{\mathbb{1}\oplus\mathcal{C}'}$. Indeed, if γ is finite then $\mathbb{1}\oplus\mathcal{C}' = \mathcal{C}$, while if γ is infinite then $\mathbb{1}\oplus\mathcal{C}' = \mathbb{1}\oplus\mathcal{C}$ so that Lemma 4.6.2 yields $\mathcal{B} \to (\mathcal{A})_k^{\mathcal{C}}$ for the same SD-barrier $\mathcal{B}$.

We thus need to prove $\mathcal{B} \to (\mathcal{A})_k^{\mathbb{1}\oplus\mathcal{C}'}$. Let s be $(1 \oplus \mathcal{B})$-size and let $c\colon [s]^{\mathbb{1}\oplus\mathcal{C}'} \to k$ be any coloring. We need to prove that there exists a homogeneous $(\mathbb{1}\oplus\mathcal{C}'\oplus\mathcal{A})$-size set for c. Let $t \subseteq s$ be the $(\mathbb{1}\oplus\mathcal{C}'\oplus\mathcal{B}(\mathcal{A}))$-size set which is $\mathbb{1}$-prehomogeneous for c. By definition $t = t_0'{}^\frown t_1'$ for some $\mathcal{B}(\mathcal{A})$-size t_0' and some $(\mathbb{1}\oplus\mathcal{C}')$-size t_1' and so we may also write $t = t_0'{}^\frown\langle\min t_1'\rangle^\frown t_1'^{-}$. We define $t_0 = t_0'{}^\frown\langle\min t_1'\rangle$ and $t_1 = t_1'^{-}$: in this way $t = t_0{}^\frown t_1$ where t_0 is $(\mathbb{1}\oplus\mathcal{B}(\mathcal{A}))$-size and t_1 is $(\mathbb{1}\oplus\mathcal{C}'_{\max t_0})$-size.

We claim that for each $n \in t_0$, $\langle n\rangle^\frown t_1$ is $(\mathbb{1}\oplus\mathcal{C}')$-large and so for each $n \in t_0$ there exists $r \subseteq t$ such that $\langle n\rangle^\frown r$ is $(\mathbb{1}\oplus\mathcal{C}')$-size. Suppose that for some n, $\langle n\rangle^\frown t_1$ is $(\mathbb{1}\oplus\mathcal{C}')$-small and let ν be a $(\mathbb{1}\oplus\mathcal{C}')$-size set such that $\langle n\rangle^\frown t_1 \sqsubset \nu$. Then since $n \leqslant \max t_0$ it follows that $\langle\max t_0\rangle^\frown t_1$ and ν contradict the smoothness of $\mathbb{1}\oplus\mathcal{C}'$.

Let $\bar{c}\colon t_0 \to k$ be the coloring defined as $\bar{c}(n) = c(\langle n\rangle^\frown r)$ where r is any $(\mathbb{1}\oplus\mathcal{C}'_n)$-size subset of t. By the claim above and since t is $\mathbb{1}$-prehomogeneous for c, $\bar{c}$ is well defined. We obtain that $\bar{c}$ is a k-coloring of a $(\mathbb{1}\oplus\mathcal{B}(\mathcal{A}))$-size set and by Theorem 4.4.5 there exists $u \in [t_0]^{\mathbb{1}\oplus\mathcal{A}}$ $\bar{c}$-homogeneous, say of color $i < k$.

We claim that $u^\frown t_1$ is a $(\mathbb{1}\oplus\mathcal{C}'\oplus\mathcal{A})$-large homogeneous set for c. We start by showing the largeness. Since u^* is $\mathcal{A}$-size by definition, we only need to prove that $\langle\max u\rangle^\frown t_1$ is $(\mathbb{1}\oplus\mathcal{C}')$-large. Towards a contradiction assume that $\langle\max u\rangle^\frown t_1$ is $(\mathbb{1}\oplus\mathcal{C}')$-small and let ν be a $(\mathbb{1}\oplus\mathcal{C}')$-size set such that $\langle\max u\rangle^\frown t_1 \sqsubset \nu$. Then since $\langle\max t_0\rangle^\frown t_1 = t_1'$ is $(\mathbb{1}\oplus\mathcal{C}')$-size and $\max u \leqslant \max t_0$, we get that ν and t_1' contradict the smoothness of $\mathbb{1}\oplus\mathcal{C}'$. Therefore $\langle\max u\rangle^\frown t_1$ is $(\mathbb{1}\oplus\mathcal{C}')$-large. We are left to prove that $u^\frown t_1$ is c-homogeneous. Since

$\langle \max t_0 \rangle ^\frown t_1$ is $(\mathbb{1} \oplus \mathcal{C}')$-size then t_1 must be $(\mathbb{1} \oplus \mathcal{C}')$-small. For this reason a $(\mathbb{1} \oplus \mathcal{C}')$-size subset w of $u ^\frown t_1$ cannot be entirely contained in t_1 and this means that $\min w \in u$. By definition $c(w) = \bar{c}(\min w) = i$.

Therefore $u ^\frown t_1$ is c-homogeneous and so s has a homogeneous $(\mathbb{1} \oplus \mathcal{C}' \oplus \mathcal{A})$-size set included in $u ^\frown t_1$. □

Notice that the above proof is modular: if Theorem 4.6.4 holds when $\gamma' < \gamma$ then Theorem 4.6.1 holds when $\gamma' < \gamma$ as well. Thus we can prove Theorem 4.6.4 by transfinite induction on γ using as hypothesis also Theorem 4.6.1 for smaller ordinals.

The next lemma deals with the case $\gamma = 1$, so that $\varphi_{\log \gamma} = \varphi_0$ is ordinal exponentiation with base ω.

Lemma 4.6.5. *Let* $k \in \mathbb{N}$ *and* $\mathcal{A}$ *be a* SD*-barrier with* $\mathrm{ht}(\mathcal{A}) = \alpha \geqslant \omega$. *There exists a* SD*-barrier* $\mathcal{B}$ *with* $\mathrm{base}(\mathcal{B}) = \mathrm{base}(\mathcal{A})$, $\mathrm{ht}(\mathcal{B}) \leqslant \omega^\alpha$ *and such that* $\mathcal{B} \to (\mathcal{A})^{2 \to \mathbb{1}}_k$.

Proof. We proceed by induction on α. For each n let $\mathcal{B}'_n$ be a SD-barrier with $\mathrm{base}(\mathcal{B}'_n) = \mathrm{base}(\mathcal{A}_n)$ and $\mathrm{ht}(\mathcal{B}'_n) \leqslant \omega^{\mathrm{ht}(\mathcal{A}_n)}$ such that $\mathcal{B}'_n \to (\mathcal{A}_n)^{2 \to \mathbb{1}}_k$. If $\mathrm{ht}(\mathcal{A}_n)$ is infinite then $\mathcal{B}'_n$ exists by induction hypothesis. If $\mathrm{ht}(\mathcal{A}_n) = 0$ then $\mathcal{A}_n$ is the degenerate front and $\mathcal{B}'_n = \mathbb{1}$ works, so in this case $\mathrm{ht}(\mathcal{B}'_n) = 1 = \omega^0 = \omega^{\mathrm{ht}(\mathcal{A}_n)}$. If $0 < \mathrm{ht}(\mathcal{A}_n) < \omega$ then by classical finite Ramsey theory there exists $m \in \mathbb{N}$ such that $\mathcal{B}'_n$ can be taken as the SD-barrier of m-tuples so in this case $\mathrm{ht}(\mathcal{B}'_n) = m < \omega \leqslant \omega^{\mathrm{ht}(\mathcal{A}_n)}$.

Let $\mathcal{B}_n$ be a SD-barrier (whose existence is proved in Section 4.4) of height $\mathrm{ht}(\mathcal{B}'_n) \otimes k$ and such that $\mathcal{B}_n \to (\mathcal{B}'_n)^{\mathbb{1}}_k$. We define $\mathcal{B} = \{s : s^- \in \mathcal{B}_{\min s}\}$ (which justifies the name $\mathcal{B}_n$ for the previous SD-barriers). Such $\mathcal{B}$ is a block but not necessarily a SD-barrier. However by Lemma 1.4.2 and by Corollary 4.1.26 there exists a SD-barrier with same height and same base of $\mathcal{B}$ and such that each of its elements is $\mathcal{B}$-large. Thus we may assume that $\mathcal{B}$ is a SD-barrier and we claim that $\mathcal{B} \to (\mathcal{A})^{2 \to \mathbb{1}}_k$.

Let s be $(\mathbb{1} \oplus \mathcal{B})$-size, $n = \min s$ and $c \colon [s]^2 \to k$. We show that there exist $t' \in [s^-]^{\mathbb{1} \oplus \mathcal{B}'_n}$ and $i < k$ such that $c(\langle n, m \rangle) = i$ for all $m \in t'$. In order to do that, we define a new coloring $\bar{c} \colon s^- \to k$ as $\bar{c}(m) = c(\langle n, m \rangle)$. Since s^- is $(\mathbb{1} \oplus \mathcal{B}_n)$-size and $\mathcal{B}_n \to (\mathcal{B}'_n)^{\mathbb{1}}_k$, there exists a $\bar{c}$-homogeneous $(\mathbb{1} \oplus \mathcal{B}'_n)$-size set, which is the desired t'.

By inductive hypothesis and since $\mathcal{B}'_n \to (\mathcal{A}_n)^{2 \to \mathbb{1}}_k$ there exists a set $t'' \in [t']^{2 \oplus \mathcal{A}_n}$ which is $\mathbb{1}$-prehomogeneous for c. Therefore $\langle n \rangle ^\frown t'' \in [s]^{2 \oplus \mathcal{A}}$ is $\mathbb{1}$-prehomogeneous for c.

Notice that

$$\begin{aligned}\operatorname{ht}(\mathcal{B}) &= \sup_{n\in\operatorname{base}(\mathcal{B})} (\operatorname{ht}(\mathcal{B}_n)+1)\\ &= \sup_{n\in\operatorname{base}(\mathcal{B})} ((\operatorname{ht}(\mathcal{B}'_n) \otimes k)+1)\\ &\leqslant \sup_{n\in\operatorname{base}(\mathcal{B})} ((\omega^{\operatorname{ht}(\mathcal{A}_n)} \otimes k)+1) \leqslant \omega^{\alpha}\end{aligned}$$

as required. □

The next Lemma deals with the case $\gamma = \omega^{\delta}$.

Lemma 4.6.6. *Assume that Theorem 4.6.1 holds for each $\gamma' < \omega^{\delta}$. Let $k \in \mathbb{N}$ and $\mathcal{A}$ and $\mathcal{C}$ be* SD-*barriers where* $\operatorname{ht}(\mathcal{A}) = \alpha$, $\operatorname{ht}(\mathcal{C}) = \omega^{\delta} \geqslant \omega$ *and* $\operatorname{base}(\mathcal{A})$ *is a final segment of* $\operatorname{base}(\mathcal{C})$. *There exists a* SD-*barrier* $\mathcal{B}$ *with* $\operatorname{base}(\mathcal{B}) = \operatorname{base}(\mathcal{A})$, $\operatorname{ht}(\mathcal{B}) \leqslant \varphi_{\delta}(\alpha)$ *and such that* $\mathcal{B} \to (\mathcal{A})_k^{\mathbb{1}\oplus\mathcal{C}\to\mathbb{1}}$.

Proof. We proceed by induction on α. If $\alpha = 0$ then $\mathcal{A}$ is the degenerate front and $\mathbb{1}\oplus\mathcal{C}\oplus\mathcal{A} = \mathbb{1}\oplus\mathcal{C}$. Take $\mathcal{B} = \mathcal{C}$: then $\mathcal{B} \to (\mathcal{A})_k^{\mathbb{1}\oplus\mathcal{C}\to\mathbb{1}}$ and $\operatorname{ht}(\mathcal{B}) = \varphi_0(\delta) \leqslant \varphi_0(\varphi_\delta(0)) = \varphi_\delta(0)$ as required.

Suppose $\alpha > 0$ and, by inductive hypothesis, for each n let $\mathcal{B}'_n$ be a SD-barrier with $\operatorname{base}(\mathcal{B}'_n) = \operatorname{base}(\mathcal{A}_n)$ such that $\operatorname{ht}(\mathcal{B}'_n) \leqslant \varphi_\delta(\operatorname{ht}(\mathcal{A}_n))$ and $\mathcal{B}'_n \to (\mathcal{A}_n)_k^{\mathbb{1}\oplus\mathcal{C}\to\mathbb{1}}$. Let $\mathcal{B}_n$ be a SD-barrier (whose existence is ensured by the hypothesis on Theorem 4.6.1 and the fact that $\operatorname{ht}(\mathcal{C}_n) < \omega^{\delta}$ for each n) such that $\operatorname{ht}(\mathcal{B}_n) \leqslant \varphi_{\log\operatorname{ht}(\mathcal{C}_n)}(\operatorname{ht}(\mathcal{B}'_n) \otimes k)$ and $\mathcal{B}_n \to (\mathcal{B}'_n)_k^{\mathbb{1}\oplus\mathcal{C}_n}$. We define $\mathcal{B} = \{s : s^- \in \mathcal{B}_{\min s}\}$ (which justifies the name $\mathcal{B}_n$ for the previous SD-barriers). As in the proof of Lemma 4.6.5 we may assume that $\mathcal{B}$ is a SD-barrier and we claim that $\mathcal{B} \to (\mathcal{A})_k^{\mathbb{1}\oplus\mathcal{C}\to\mathbb{1}}$.

Let s be $(\mathbb{1}\oplus\mathcal{B})$-size, $n = \min s$ and $c\colon [s]^{\mathbb{1}\oplus\mathcal{C}} \to k$. We show that there exist $t' \in [s^-]^{\mathbb{1}\oplus\mathcal{B}'_n}$ and $i < k$ such that $c(\langle n\rangle^\frown r) = i$ for all $r \in [t']^{\mathbb{1}\oplus\mathcal{C}_n}$. We define $\bar{c}\colon [s^-]^{\mathbb{1}\oplus\mathcal{C}_n} \to k$ as $\bar{c}(r) = c(\langle n\rangle^\frown r)$. Since s^- is $(\mathbb{1}\oplus\mathcal{B}_n)$-size and $\mathcal{B}_n \to (\mathcal{B}'_n)_k^{\mathbb{1}\oplus\mathcal{C}_n}$ then there exists a $\bar{c}$-homogeneous $(\mathbb{1}\oplus\mathcal{C}_n\oplus\mathcal{B}'_n)$-size set. Its $(\mathbb{1}\oplus\mathcal{B}'_n)$-size prefix is the desired t'.

Since $\mathcal{B}'_n \to (\mathcal{A}_n)_k^{\mathbb{1}\oplus\mathcal{C}\to\mathbb{1}}$, there exists a set $t'' \in [t']^{\mathbb{1}\oplus\mathcal{C}\oplus\mathcal{A}_n}$ which is $\mathbb{1}$-prehomogeneous for c. Therefore $\langle n\rangle^\frown t'' \in [s]^{\mathbb{1}\oplus\mathcal{C}\oplus\mathcal{A}}$ is $\mathbb{1}$-prehomogeneous for c.

We are left to prove that the height of $\mathcal{B}$ is bounded by $\varphi_\delta(\alpha)$. Indeed

$$\begin{aligned}\operatorname{ht}(\mathcal{B}) &= \sup_{n\in\operatorname{base}(\mathcal{B})} (\operatorname{ht}(\mathcal{B}_n)+1)\\ &\leqslant \sup_{n\in\operatorname{base}(\mathcal{B})} (\varphi_{\log\operatorname{ht}(\mathcal{C}_n)}(\operatorname{ht}(\mathcal{B}'_n) \otimes k)+1)\\ &\leqslant \sup_{n\in\operatorname{base}(\mathcal{B})} (\varphi_{\log\operatorname{ht}(\mathcal{C}_n)}(\varphi_\delta(\operatorname{ht}(\mathcal{A}_n)) \otimes k)+1)\\ &\leqslant \sup_{n\in\operatorname{base}(\mathcal{B})} (\varphi_{\log\operatorname{ht}(\mathcal{C}_n)}(\varphi_\delta(\alpha))) = \varphi_\delta(\alpha)\end{aligned}$$

where the last inequality follows because the Veblen functions are strictly increasing, while the last equality follows by the fact that for every n $\varphi_{\log \mathrm{ht}(\mathcal{C}_n)}$ is a composition of Veblen functions each of index strictly smaller than δ and so $\varphi_\delta(\mathrm{ht}(\mathcal{A}_s))$ is a fixed point of $\varphi_{\log \mathrm{ht}(\mathcal{C}_n)}$. □

The next step is to prove a strengthening of Lemma 4.6.6 which is useful for the proof of the remaining cases of Theorem 4.6.4.

Lemma 4.6.7. *Assume that Theorem 4.6.1 holds for each* $\gamma' < \omega^\delta$. *Let* $k \in \mathbb{N}$ *and* $\mathcal{A}$, $\mathcal{C}$ *and* $\mathcal{D}$ *be* SD-barriers *where* $\mathrm{ht}(\mathcal{A}) = \alpha$, $\mathrm{ht}(\mathcal{C}) = \omega^\delta$ *and* $\mathrm{base}(\mathcal{A})$ *and* $\mathrm{base}(\mathcal{D})$ *are final segments of* $\mathrm{base}(\mathcal{C})$. *There exists a* SD-barrier $\mathcal{B}$ *with* $\mathrm{ht}(\mathcal{B}) \leqslant \varphi_\delta(\alpha)$ *and* $\mathrm{base}(\mathcal{B}) = \mathrm{base}(\mathcal{A})$ *such that* $\mathcal{B} \to (\mathcal{A})_k^{\mathbb{1}\oplus\mathcal{C}\oplus\mathcal{D}\to\mathbb{1}\oplus\mathcal{D}}$.

Proof. Starting from $\mathcal{A}$ and $\mathcal{C}$ we define sets of SD-barriers $(\mathcal{F}_s)_{s\in T(\mathcal{A})}$ and $(\mathcal{E}_s)_{s\in T(\mathcal{A})}$ where $\mathrm{base}(\mathcal{E}_s) = \mathrm{base}(\mathcal{F}_s) = \mathrm{base}(\mathcal{A}) \setminus \{0,\dots,\max s\}$. If $s \in \mathcal{A}$ then we define $\mathcal{F}_s = \mathcal{C}$ restricted to $\mathrm{base}(\mathcal{A}) \setminus \{0,\dots,\max s\}$. If $s \in T(\mathcal{A}) \setminus \mathcal{A}$, assume that for each $n > \max s$ we have defined $\mathcal{F}_{s^\frown\langle n\rangle}$. Let $\mathcal{E}_{s^\frown\langle n\rangle}$ be a SD-barrier (whose existence is ensured by the inductive assumption on Theorem 4.6.1 and by the fact that $1 + \mathrm{ht}(\mathcal{C}_n) < \omega^\delta$ for each n) with $\mathrm{base}(\mathcal{E}_{s^\frown\langle n\rangle}) = \mathrm{base}(\mathcal{F}_{s^\frown\langle n\rangle})$, $\mathrm{ht}(\mathcal{E}_{s^\frown\langle n\rangle}) \leqslant \varphi_{\log \mathrm{ht}(\mathcal{C}_n)}(\mathrm{ht}(\mathcal{F}_{s^\frown\langle n\rangle}) \otimes k^{(2^{|s|})})$ and $\mathcal{E}_{s^\frown\langle n\rangle} \to (\mathcal{F}_{s^\frown\langle n\rangle})^{\mathbb{1}\oplus\mathcal{C}_n}_{k^{(2^{|s|})}}$. We define $\mathcal{F}_s$ as the set $\{t \subseteq \mathrm{base}(\mathcal{A}) : t^- \in \mathcal{E}_{s^\frown\langle \min t\rangle}\}$. Then $\mathcal{F}_s$ is clearly a block with $\mathrm{base}(\mathcal{F}_s) = \mathrm{base}(\mathcal{A}) \setminus \{0,\dots,\max s\}$. By Lemma 1.4.2 and Corollary 4.1.26 we may assume that $\mathcal{F}_s$ is a SD-barrier with same height and same base.

We claim that for each $s \in T(\mathcal{A})$, $\mathrm{ht}(\mathcal{F}_s) \leqslant \varphi_\delta(\mathrm{ht}(\mathcal{A}_s))$. We proceed by induction on $\mathrm{ht}(\mathcal{A}_s)$. If $s \in \mathcal{A}$ then $\mathcal{F}_s = \mathcal{C}$ and $\mathrm{ht}(\mathcal{F}_s) = \omega^\delta \leqslant \varphi_\delta(0) = \varphi_\delta(\mathrm{ht}(\mathcal{A}_s))$. Now suppose that $s \in T(\mathcal{A}) \setminus \mathcal{A}$ and the claim holds for all $s^\frown\langle n\rangle \in T(\mathcal{A})$. By construction, $\mathrm{ht}(\mathcal{E}_{s^\frown\langle n\rangle}) \leqslant \varphi_{\log \mathrm{ht}(\mathcal{C}_n)}(\mathrm{ht}(\mathcal{F}_{s^\frown\langle n\rangle}) \otimes k^{(2^{|s|})})$. It follows that

$$\begin{aligned}
\mathrm{ht}(\mathcal{F}_s) &= \sup_{n\in\mathrm{base}(\mathcal{F}_s)} (\mathrm{ht}(\mathcal{E}_{s^\frown\langle n\rangle}) + 1) \\
&\leqslant \sup_{n\in\mathrm{base}(\mathcal{F}_s)} (\varphi_{\log \mathrm{ht}(\mathcal{C}_n)}(\mathrm{ht}(\mathcal{F}_{s^\frown\langle n\rangle}) \otimes k^{(2^{|s|})}) + 1) \\
&\leqslant \sup_{n\in\mathrm{base}(\mathcal{F}_s)} (\varphi_{\log \mathrm{ht}(\mathcal{C}_n)}(\varphi_\delta(\mathrm{ht}(\mathcal{A}_{s^\frown\langle n\rangle})) \otimes k^{(2^{|s|})}) + 1) \\
&\leqslant \sup_{n\in\mathrm{base}(\mathcal{F}_s)} \varphi_{\log(\mathrm{ht}(\mathcal{C}_n))}(\varphi_\delta(\mathrm{ht}(\mathcal{A}_s))) = \varphi_\delta(\mathrm{ht}(\mathcal{A}_s))
\end{aligned}$$

where the last inequality follows since the Veblen functions are strictly increasing and the last equality follows by the fact that for every n $\varphi_{\log \mathrm{ht}(\mathcal{C}_n)}$ is a composition of Veblen functions each of index strictly smaller than δ and so $\varphi_\delta(\mathrm{ht}(\mathcal{A}_s))$ is a fixed point of $\varphi_{\log \mathrm{ht}(\mathcal{C}_n)}$.

In particular $\mathrm{ht}(\mathcal{F}_{\langle\rangle}) \leqslant \varphi_\delta(\alpha)$, so we set $\mathcal{B} = \mathcal{F}_{\langle\rangle}$ and we prove $\mathcal{B} \to (\mathcal{A})_k^{\mathbb{1}\oplus\mathcal{C}\oplus\mathcal{D}\to\mathbb{1}\oplus\mathcal{D}}$.

Let s be $(\mathbb{1}\oplus\mathcal{B})$-size and let $c\colon [s]^{\mathbb{1}\oplus\mathcal{C}\oplus\mathcal{D}} \to k$. Using an idea similar to Mathias forcing, we recursively construct sequences $f_0, f_1, \ldots \in s$ and $F_0, F_1, \ldots \subseteq s$ such that for each n (letting $u_n = \langle f_0, \ldots, f_n\rangle$) the following holds:

(1) $f_0 < f_1 < \ldots < f_n < F_n$,

(2) $f_n = \min F_{n-1}$,

(3) $F_n \subseteq F_{n-1}$,

(4) F_n is $(1 \oplus \mathcal{F}_{u_n})$-large,

(5) let $t_1, t_2 \in [s]^{\mathbb{1}\oplus\mathcal{C}\oplus\mathcal{D}}$ and let t_1' and t_2' be their $(\mathbb{1}\oplus\mathcal{D})$-size prefixes: if $t_1' = t_2' \subseteq u_n$ and $t_1 \setminus t_1', t_2 \setminus t_2' \subseteq F_n$ then $c(t_1) = c(t_2)$.

We initialize the construction by setting $F_{-1} = s$ and leaving f_{-1} undefined, so that $u_{-1} = \langle\rangle$. Clauses $1 \div 4$ are trivial. Since $(\mathbb{1}\oplus\mathcal{C}\oplus\mathcal{D})$-size sets have at least two elements, clause 5 is trivial too.

Suppose we are at stage n and that we have already defined f_{n-1} and F_{n-1} satisfying clauses $1 \div 5$. Let $f_n = \min F_{n-1}$ (so that clause 2 is satisfied) and let $E_n = F_{n-1}^-$. Let $c_n'\colon [E_n]^{\mathbb{1}\oplus\mathcal{C}_{f_n}} \to k^{(2^n)}$ be the function that maps $e \in [E_n]^{\mathbb{1}\oplus\mathcal{C}_{f_n}}$ to (a code for) the function $[u_{n-1}]^{\mathcal{D}} \to k$ defined by $d \mapsto c(d^\frown\langle f_n\rangle^\frown e)$. Since u_{n-1} has 2^n subsets, then $[u_{n-1}]^{\mathcal{D}}$ has at most 2^n elements. Therefore there are at most $k^{(2^n)}$ functions $[u_{n-1}]^{\mathcal{D}} \to k$ and so c_n' is well defined. We know that F_{n-1} is $(1 \oplus \mathcal{F}_{u_{n-1}})$-large and so F_{n-1}^* is $\mathcal{F}_{u_{n-1}}$-large. It follows that $E_n^* = F_{n-1}^{*-}$ is $\mathcal{E}_{u_n}$-large and hence E_n is $(\mathbb{1}\oplus\mathcal{E}_{u_n})$-large. By construction $\mathcal{E}_{u_n} \to (\mathcal{F}_{u_n})_{k^{(2^n)}}^{\mathbb{1}\oplus\mathcal{C}_{f_n}}$ and so there is a $(\mathbb{1}\oplus\mathcal{C}_{f_n}\oplus\mathcal{F}_{u_n})$-large subset of E_n which is c_n'-homogeneous. We call such set F_n and notice that F_n is $(\mathbb{1}\oplus\mathcal{F}_{u_n})$-large (hence clauses 3 and 4 are satisfied). Since $F_n \subseteq E_n$, $f_n < F_n$ and clause 1 holds too. We are left to prove that clause 5 is satisfied. Let $t_1, t_2 \in [s]^{\mathbb{1}\oplus\mathcal{C}\oplus\mathcal{D}}$ and let t_1' and t_2' be their $(\mathbb{1}\oplus\mathcal{D})$-size prefixes. Moreover assume that $t_1' = t_2' \subseteq u_n$ and $t_1 \setminus t_1', t_2 \setminus t_2' \subseteq F_n$. If $t_1' = t_2' \subseteq u_{n-1}$ then since $F_n \subseteq F_{n-1}$, by clause 5 of stage $n-1$ we are done. If not then $t_1'^* = t_2'^* \in [u_{n-1}]^{\mathcal{D}}$ and $\max t_1' = \max t_n' = f_n$. Since F_n is c_n'-homogeneous, then $d \mapsto c(d^\frown\langle f_n\rangle^\frown t_1 \setminus t_1')$ and $d \mapsto c(d^\frown\langle f_n\rangle^\frown t_2 \setminus t_2')$ are the same function (notice that $t_1 \setminus t_1'$ and $t_2 \setminus t_2'$ are $(\mathbb{1}\oplus\mathcal{C}_{f_n})$-size). Therefore $c(t_1) = c(t_1'^{*\frown}\langle f_n\rangle^\frown t_1 \setminus t_1') = c(t_2'^{*\frown}\langle f_n\rangle^\frown t_2 \setminus t_2') = c(t_2)$.

The construction stops at a stage m such that $u_m \in \mathcal{A}$. At this stage $\mathcal{F}_{u_m} = \mathcal{C}$ by definition and so F_m is $(\mathbb{1}\oplus\mathcal{C})$-large. Let $w \sqsubseteq F_m$ be $(\mathbb{1}\oplus\mathcal{C})$-size and let $f_{m+1} = \min w$. The set $v = u_m{}^\frown w$ is $(\mathbb{1}\oplus\mathcal{C}\oplus\mathcal{A})$-size. Let $t_1, t_2 \in [v]^{\mathbb{1}\oplus\mathcal{C}\oplus\mathcal{D}}$ and let t_1' and t_2' be their $(\mathbb{1}\oplus\mathcal{D})$-size prefixes. Moreover assume that $t_1' = t_2'$. Suppose first that $\max t_1' = \max t_2' \geqslant f_{m+1}$ then it must be $t_1 \setminus t_1'^*, t_2 \setminus t_2'^* \in [w]^{\mathbb{1}\oplus\mathcal{C}}$ and since $\mathbb{1}\oplus\mathcal{C}$ is a barrier we get $t_1 \setminus t_1'^* = w = t_2 \setminus t_2'^*$.

Hence $t_1 = t_2$ and $c(t_1) = c(t_2)$. Suppose now that $\max t'_1 = \max t'_2 = f_n$ for some $n \leqslant m$ and so $t'^*_1 = t'^*_2 \in [u_{n-1}]^{\mathcal{D}}$. Since F_n is c'_n-homogeneous and $\langle f_{n+1}, \dots, f_m\rangle^\frown w \subseteq F_n$, then $d \mapsto c(d^\frown\langle f_n\rangle^\frown t_1 \setminus t'_1)$ and $d \mapsto c(d^\frown\langle f_n\rangle^\frown t_2 \setminus t'_2)$ are the same function. Therefore

$$c(t_1) = c(t'^{*\frown}_1\langle f_n\rangle^\frown t_1 \setminus t'_1) = c(t'^{*\frown}_2\langle f_n\rangle^\frown t_2 \setminus t'_2) = c(t_2).$$

Hence v is a $(\mathbb{1} \oplus \mathcal{C} \oplus \mathcal{A})$-size $(1 \oplus \mathcal{D})$-prehomogeneous set for c. □

We are finally ready to prove Theorem 4.6.4, concluding Section 4.6.

Proof of Theorem 4.6.4. Recall that Theorem 4.6.4 states the following: given $k \in \mathbb{N}$ and $\mathcal{A}$ and $\mathcal{C}$ SD-barriers where $\mathrm{ht}(\mathcal{A}) = \alpha$, $\mathrm{ht}(\mathcal{C}) = \gamma$ and $\mathrm{base}(\mathcal{A})$ is a final segment of $\mathrm{base}(\mathcal{C})$, there exists a SD-barrier $\mathcal{B}$ with $\mathrm{ht}(\mathcal{B}) \leqslant \varphi_{\log\gamma}(\alpha)$ and the same base as $\mathcal{A}$ such that $\mathcal{B} \to (\mathcal{A})^{\mathbb{1}\oplus\mathcal{C}\to\mathbb{1}}_k$.

We proceed by induction on γ. If $\gamma = 0$ then $\mathcal{B} = \mathcal{A}$ works. If $\gamma > 0$ the induction hypothesis and the modularity of the proof that Theorem 4.6.4 implies Theorem 4.6.1 yield that Theorem 4.6.4 holds for every $\gamma' < \gamma$.

The case $\gamma = \omega^\delta$ is proved in Lemmas 4.6.5 and 4.6.6.

Suppose now that $\gamma = \omega^\delta + \nu$ with $\omega^\delta \ggg \nu > 0$. Since $\mathcal{C}$ is decomposable, there exist SD-barriers $\mathcal{C}_0$, $\mathcal{C}_1$ such that $\mathrm{ht}(\mathcal{C}_0) = \omega^\delta$, $\mathrm{ht}(\mathcal{C}_1) = \nu$ and $\mathcal{C} = \mathcal{C}_0 \oplus \mathcal{C}_1$. By inductive hypothesis (since $\mathrm{ht}(\mathcal{C}_1) = \nu < \gamma$) there exists a SD-barrier $\mathcal{B}'$ with $\mathrm{ht}(\mathcal{B}') \leqslant \varphi_{\log\nu}(\alpha)$ and $\mathrm{base}(\mathcal{B}') = \mathrm{base}(\mathcal{A})$ such that $\mathcal{B}' \to (\mathcal{A})^{\mathbb{1}\oplus\mathcal{C}_1\to\mathbb{1}}_k$. By Lemma 4.6.7 there exists a SD-barrier $\mathcal{B}$ with $\mathrm{ht}(\mathcal{B}) \leqslant \varphi_\delta(\varphi_{\log\nu}(\alpha)) = \varphi_{\log\gamma}(\alpha)$ and $\mathrm{base}(\mathcal{B}) = \mathrm{base}(\mathcal{B}')$ such that $\mathcal{B} \to (\mathcal{B}')^{\mathbb{1}\oplus\mathcal{C}_0\oplus\mathcal{C}_1\to\mathbb{1}\oplus\mathcal{C}_1}_k$. We claim that $\mathcal{B} \to (\mathcal{A})^{\mathbb{1}\oplus\mathcal{C}\to\mathbb{1}}_k$.

Let s be $(\mathbb{1}\oplus\mathcal{B})$-size and $c\colon [s]^{\mathbb{1}\oplus\mathcal{C}} \to k$ be a coloring. Since $\mathcal{C} = \mathcal{C}_0 \oplus \mathcal{C}_1$ and $\mathcal{B} \to (\mathcal{B}')^{\mathbb{1}\oplus\mathcal{C}_0\oplus\mathcal{C}_1\to\mathbb{1}\oplus\mathcal{C}_1}_k$, there exists $t \in [s]^{\mathbb{1}\oplus\mathcal{C}_0\oplus\mathcal{B}'}$ which is $(\mathbb{1}\oplus\mathcal{C}_1)$-prehomogeneous for c. Let $t = t'^\frown_0 t'_1$ where t'_0 is $\mathcal{B}'$-size t'_0 and t'_1 is $(\mathbb{1}\oplus\mathcal{C}_0)$-size. We may also write $t = t'^\frown_0\langle\min t'_1\rangle^\frown t'^-_1$. We define $t_0 = t'^\frown_0\langle\min t'_1\rangle$ and $t_1 = t'^-_1$ so that $t = t_0^\frown t_1$, t_0 is $(\mathbb{1}\oplus\mathcal{B}')$-size and t_1 is $(\mathbb{1}\oplus(\mathcal{C}_0)_{\max t_0})$-size.

Let $\bar{c}\colon [t_0]^{\mathbb{1}\oplus\mathcal{C}_1} \to k$ be the coloring defined as $\bar{c}(d) = c(d^\frown r)$ where r is any $(\mathbb{1}\oplus(\mathcal{C}_0)_{\max d})$-size subset of t. We claim that for each $d \in [t_0]^{\mathbb{1}\oplus\mathcal{C}_1}$, $d^\frown t_1$ is $(\mathbb{1}\oplus\mathcal{C})$-large and so there exists $r \subseteq t$ which is $(\mathbb{1}\oplus(\mathcal{C}_0)_{\max d})$-size. Suppose $d^\frown t_1$ is $(\mathbb{1}\oplus\mathcal{C})$-small for some $(\mathbb{1}\oplus\mathcal{C}_1)$-size d. This implies that $\langle\max d\rangle^\frown t_1$ is $(\mathbb{1}\oplus\mathcal{C}_0)$-small. Let v be a $(\mathbb{1}\oplus\mathcal{C}_0)$-size set such that $\langle\max d\rangle^\frown t_1 \sqsubset v$. From $\max d \leqslant \max t_0$ it follows that $\langle\max t_0\rangle^\frown t_1$ and v contradict the smoothness of $\mathbb{1}\oplus\mathcal{C}_0$. Moreover, since t is $(\mathbb{1}\oplus\mathcal{C}_1)$-prehomogeneous for c, $\bar{c}$ is well defined. We obtain that $\bar{c}$ is a coloring of a $(\mathbb{1}\oplus\mathcal{B}')$-size set and since $\mathcal{B}' \to (\mathcal{A})^{\mathbb{1}\oplus\mathcal{C}_1\to\mathbb{1}}_k$ there exists $u \in [t_0]^{\mathbb{1}\oplus\mathcal{C}_1\oplus\mathcal{A}}$ which is $\mathbb{1}$-prehomogeneous for $\bar{c}$.

We claim that $u^\frown t_1$ is a $(\mathbb{1} \oplus \mathcal{C} \oplus \mathcal{A})$-large $\mathbb{1}$-prehomogeneous set for c. We start by showing the largeness. Let u' be the $\mathcal{A}$-size initial segment of u. We need to prove that $u \setminus u'^\frown t_1$ is $(\mathbb{1} \oplus \mathcal{C})$-large. Notice that $u \setminus u'$ is $(\mathbb{1} \oplus \mathcal{C}_1)$-size. Towards a contradiction assume that $u \setminus u'^\frown t_1$ is $(\mathbb{1} \oplus \mathcal{C})$-small. In other words suppose that $\langle \max u \rangle^\frown t_1$ is $(\mathbb{1} \oplus \mathcal{C}_0)$-small. Let v be a $(\mathbb{1} \oplus \mathcal{C}_0)$-size set such that $\langle \max u \rangle^\frown t_1 \sqsubset v$. Since $\langle \max t_0 \rangle^\frown t_1 = t_1'$ is $(\mathbb{1} \oplus \mathcal{C}_0)$-size and $\max u \leqslant \max t_0$, we get that v and t_1' contradict the smoothness of $\mathbb{1} \oplus \mathcal{C}_0$. Therefore $u \setminus u'^\frown t_1$ is $(\mathbb{1} \oplus \mathcal{C})$-large and so $u^\frown t_1$ is $(\mathbb{1} \oplus \mathcal{C} \oplus \mathcal{A})$-large.

We are left to prove that $u^\frown t_1$ is $\mathbb{1}$-prehomogeneous for c. Let $w \in [u^\frown t_1]^{\mathbb{1} \oplus \mathcal{C}}$ and let $w' \sqsubset w$ be $(\mathbb{1} \oplus \mathcal{C}_1)$-size. We claim that $w' \subseteq u$. Otherwise if we write $w = w'^{*\frown} \langle \max w' \rangle^\frown w \setminus w'$, we get $\langle \max w' \rangle^\frown w \setminus w' \subseteq t_1$. Since w'^* is $\mathcal{C}_1$-size we have that $\langle \max w' \rangle^\frown w \setminus w'$ is $(\mathbb{1} \oplus \mathcal{C}_0)$-size which is a contradiction since $\langle \max w' \rangle^\frown w \setminus w' \subseteq t_1$ and t_1 is $(\mathbb{1} \oplus \mathcal{C}_0)$-small by its definition. Therefore $w' \subseteq u$. By definition of $\bar{c}$ we have $\bar{c}(w') = c(w)$ and since u is $\mathbb{1}$-prehomogeneous for $\bar{c}$ it follows that $c(w)$ depends only on $\min w$. Hence $u^\frown t_1$ is $\mathbb{1}$-prehomogeneous for c. □

Theorem 4.5.1 together with Theorem 4.6.1 proves Theorem 4.3.8.

4.7 NESTEDNESS OF THE SYSTEM

This section is devoted to the proof that the system of fundamental sequences of Definition 4.1.13 on Γ_ζ is nested (Theorem 4.1.14). Recall that a system of fundamental sequences is nested if it is never the case for $n > 1$ that

$$\beta > \gamma > \beta[n] > \gamma[n].$$

The first few lemmas show that, in some circumstances, the fundamental sequences for ordinals above some bound lie entirely above that bound. Each ordinal in the rest of the section is below Γ_ζ.

Lemma 4.7.1. *Let $k \in \omega$ and γ be an ordinal. If $\varphi_0^k(0) < \gamma < \epsilon_0$ then $\varphi_0^k(0) \leqslant \gamma[1]$ and hence $\varphi_0^k(0) \leqslant \gamma[n]$ for all $n > 0$.*

Proof. We proceed by induction on k. The case $k = 0$ is obvious as $\varphi_0^0(0) = 0$. Assume now that $\varphi_0^{k+1}(0) < \gamma < \epsilon_0$. If $\gamma = \omega^\alpha$ then $\varphi_0^k(0) < \alpha < \epsilon_0$ and by induction hypothesis $\varphi_0^k(0) \leqslant \alpha[1]$: then $\varphi_0^{k+1}(0) \leqslant \omega^{\alpha[1]} = \gamma[1]$. Otherwise, we can write $\gamma = \omega^\alpha + \beta$ for some $\alpha \geqslant \varphi_0^{k+1}(0)$ and $\omega^\alpha \gg \beta$. Then $\gamma[1] = \omega^\alpha + \beta[1] \geqslant \varphi_0^{k+1}(0)$. □

Lemma 4.7.2. *For every $\delta > \epsilon_0$ we have $\delta[1] \geqslant \epsilon_0$ and hence $\delta[n] \geqslant \epsilon_0$ for all $n > 0$. Similarly, for every $\delta > \Gamma_0$ we have $\delta[1] \geqslant \Gamma_0$ and hence $\delta[n] \geqslant \Gamma_0$ for all $n > 0$.*

Proof. We start with the first statement. The proof is by induction on δ. If the Cantor normal form of δ has more than one term, then $\delta[1]$ is larger or equal than the first term, which is $\geqslant \epsilon_0$. If the normal form of δ consists of a single term we distinguish four cases.

- If $\delta = \omega^\alpha$ with $\delta > \alpha$ then we must have $\alpha > \epsilon_0$ and the induction hypothesis yields $\alpha[1] \geqslant \epsilon_0$. Hence $\delta[1] = \omega^{\alpha[1]} \geqslant \epsilon_0$.
- If $\delta = \varphi_{\delta_0}(0) > \delta_0$ then $\delta_0 > 1$ and we have $\delta_0[1] > 0$. Thus $\delta[1] = \varphi^2_{\delta_0[1]}(0) > \epsilon_0$.
- If $\delta = \varphi_{\delta_0}(\alpha)$ with $\delta > \alpha > 0$ and $\delta_0 > 0$ we have $\varphi_{\delta_0}(\alpha[1]) \geqslant \epsilon_0$ and, a fortiori, $\delta[1] = \varphi^2_{\delta_0[1]}(\varphi_{\delta_0}(\alpha[1]) + 1) > \epsilon_0$.
- If $\delta = \Gamma_\xi$ then, for some β depending on ξ, $\delta[1] = \varphi_{\varphi_\beta(0)}(0) \geqslant \epsilon_0$.

For the second statement we proceed similarly, treating only the cases where a different argument is needed.

- If $\delta = \varphi_{\delta_0}(0) > \delta_0$ then $\delta_0 > \Gamma_0$ and we have $\delta_0[1] \geqslant \Gamma_0$ by inductive hypothesis. Thus $\delta[1] = \varphi^2_{\delta_0[1]}(0) > \Gamma_0$.
- Suppose $\delta = \varphi_{\delta_0}(\alpha)$ with $\delta > \alpha > 0$ and $\delta_0 > 0$. We consider three subcases depending on the value of α.
  - When $\alpha = 0$ we must have $\delta_0 > \Gamma_0$; by inductive hypothesis $\delta_0[1] \geqslant \Gamma_0$ and so $\delta[1] = \varphi^2_{\delta_0[1]}(0) > \Gamma_0$.
  - When $0 < \alpha < \Gamma_0$ we must have $\delta_0 \geqslant \Gamma_0$; then we obtain $\delta[1] = \varphi^2_{\delta_0[1]}(\varphi_{\delta_0}(\alpha[1]) + 1) > \Gamma_0$.
  - Finally, if $\alpha > \Gamma_0$, by inductive hypothesis we have $\alpha[1] \geqslant \Gamma_0$, and then $\delta[1] = \varphi^2_{\delta_0[1]}(\varphi_\delta(\alpha[1]) + 1) > \Gamma_0$.
- If $\delta = \Gamma_\xi$ then $\delta[1] = \varphi_{\varphi_{\Gamma_{\xi[n]}+1}(0)}(0) > \Gamma_0$. □

Lemma 4.7.3. *Fix δ and suppose $\gamma < \beta$ are such that γ is in the range of φ_δ but β is not. Then $\gamma \leqslant \beta[1]$ and hence $\gamma \leqslant \beta[n]$ for all $n > 0$. The same holds if $\gamma < \beta$ are such that $\gamma = \Gamma_{\gamma_1}$ but β is not of the form Γ_δ for any δ.*

Proof. Consider first the case of the Veblen functions. Let $\gamma = \varphi_\delta(\gamma_1)$ and argue by induction on β. If the Cantor normal form of β has more than one term then $\beta[1]$ is larger or equal than the first term, which is $\geqslant \varphi_\delta(\gamma_1) = \gamma$. Since β is not in the range of φ_δ, the remaining case is $\beta = \varphi_\xi(\alpha)$ for some $\xi < \delta$ and $\alpha < \beta$. If $\alpha \leqslant \gamma$ then $\beta = \varphi_\xi(\alpha) \leqslant \varphi_\xi(\varphi_\delta(\gamma_1)) = \varphi_\delta(\gamma_1) = \gamma$ which is a contradiction. Thus $\gamma < \alpha$ and by the induction hypothesis (α is not a fixed point of φ_ξ and hence is not in the range of φ_δ) we have $\gamma \leqslant \alpha[1]$. Since $\beta[1] = \varphi^2_{\xi[1]}(\varphi_\xi(\alpha[1]) + 1) > \alpha[1]$, we get $\gamma < \beta[1]$.

In the case of the Γ function we again proceed by induction on β. Recall that Γ_{γ_1} is the γ_1-th ordinal such that $\varphi_\gamma(0) = \gamma$. The case β decomposable is trivial as before. We suppose $\beta = \varphi_\xi(\alpha) > \alpha$ and we distinguish different subcases.

- If $\beta = \varphi_\xi(0)$ then $\xi > \gamma$ and by inductive hypothesis $\xi[1] \geqslant \gamma$. Hence $\beta[1] = \varphi^2_{\xi[1]}(0) > \gamma$.
- If $\beta = \varphi_\xi(\alpha)$ with $0 < \alpha \leqslant \gamma$ then it must be $\xi \geqslant \gamma$. Hence $\beta[1] = \varphi^2_{\xi[1]}(\varphi_\xi(\alpha[1]) + 1) \geqslant \varphi^2_{\xi[1]}(\varphi_\gamma(\alpha[1]) + 1) > \gamma$.
- If $\beta = \varphi_\xi(\alpha)$ with $\alpha > \gamma$ then by inductive hypothesis $\alpha[1] \geqslant \gamma$. Hence $\beta[1] = \varphi^2_{\xi[1]}(\varphi_\xi(\alpha[1]) + 1) > \varphi^2_{\xi[1]}(\gamma) \geqslant \gamma$. □

Lemma 4.7.4. *Let γ and δ be ordinals such that γ is not of the form $\varphi_{\gamma_0}(0)$ for any γ_0. If $\gamma > \varphi_\delta(0)$ then $\gamma[1] \geqslant \varphi_\delta(0)$.*

Proof. If γ is decomposable or $\delta = 0$ the thesis is trivial. We can assume $\gamma = \varphi_\xi(\gamma_0) > \gamma_0 > 0$. Case $\delta = 1$ was proved in Lemma 4.7.2. Case $\xi < \delta$ follows by Lemma 4.7.3. Finally, if $\xi \geqslant \delta$ then $\gamma[1] = \varphi^2_{\xi[1]}(\varphi_\xi(\gamma_0[1]) + 1) \geqslant \varphi^2_{\xi[1]}(\varphi_\delta(\gamma_0[1]) + 1) > \varphi_\delta(0)$. □

Notice that the previous result fails when γ of the form $\varphi_{\gamma_0}(0)$: e.g. $\varphi_\omega(0) > \varphi_3(0) > \varphi^2_1(0) = \varphi_\omega(0)[1]$.

Lemma 4.7.5. *Let $n > 1$ and Γ_ξ and δ ordinals. Then $\delta = \Gamma_\xi + 1$ if and only if $\delta[n] = \Gamma_\xi$.*

Proof. The forward implication follows from the regularity of the system of fundamental sequences.

For the backward direction suppose δ is decomposable. Then, by regularity, $\delta[n] = \Gamma_\xi$ is greater or equal than the first term in the Cantor normal form of δ. Hence $\delta = \Gamma_\xi + \beta$ with $\Gamma_\xi \ggg \beta > 0$. Then $\Gamma_\xi = \delta[n] = \Gamma_\xi + \beta[n]$ implies $\beta[n] = 0$. Since $n > 0$, this can only happen if $\beta = 1$. We are left to prove that δ cannot be indecomposable and we do that by cases.

- If $\delta = \omega^{\delta_0} > \delta_0$ then $\delta[n] = \omega^{\delta_0[n]} \cdot n$. Since $n > 1$, $\delta[n]$ is decomposable contradicting $\delta[n] = \Gamma_\xi$.
- If $\delta = \varphi_{\delta_0}(0) > \delta_0$ then $\delta_0 > 0$ and $\delta[n] = \varphi^{n+1}_{\delta_0[n]}(0)$. If $\delta_0[n] < \Gamma_\xi$ then $\delta[n] < \Gamma_\xi$. If $\delta_0[n] \geqslant \Gamma_\xi$ then, since $n > 0$, $\delta[n] > \Gamma_\xi$.

- If $\delta = \varphi_{\delta_0}(\delta_1) > \delta_1 > 0$ then $\delta[n] = \varphi^{n+1}_{\delta_0[n]}(\varphi_{\delta_0}(\delta_1[n]) + 1)$. If $\delta_0[n] < \Gamma_\xi$ then Γ_ξ is a fixed point of $\varphi_{\delta_0[n]}$ and so $\Gamma_\xi = \delta[n]$ implies $\Gamma_\xi = \varphi_{\delta_0}(\delta_1[n]) + 1$ which cannot be. If $\delta_0[n] \geqslant \Gamma_\xi$ then $\delta[n] > \Gamma_\xi$.

- If $\delta = \Gamma_\alpha$ then $\alpha > \xi$ otherwise $\Gamma_\xi \geqslant \delta > \delta[n]$. In particular $\alpha > 0$. Then if $\Gamma_\xi = \delta[n]$, since $\Gamma_\xi = \varphi_{\Gamma_\xi}(0)$ and by the definition of $\Gamma_\alpha[n]$, we get $\Gamma_\xi = \Gamma_{\alpha[n]} + 1$ which cannot be. □

Proof of Theorem 4.1.14. We prove that for every $\beta, \gamma < \Gamma_\zeta$ and $n > 1$:

(A) it is never the case that $\gamma[n] < \beta[n] < \gamma < \beta$

(B) for every $\delta < \Gamma_\zeta$, if $\gamma < \varphi_\delta(\gamma)$ it is not the case that $\gamma[n] < \varphi_\delta(\beta[n]) < \gamma < \varphi_\delta(\beta)$.

Remark 4.7.6. Notice that the hypothesis $\gamma < \varphi_\delta(\gamma)$ in (B) is necessary: let λ be a limit ordinal, $\delta = \lambda[n]$, $\beta = \varphi_\lambda(0)$ and $\gamma = \varphi_{\delta+1}(0)$. Then $\beta[n] = \gamma[n] = \varphi^{n+1}_\delta(0) < \varphi^{n+2}_\delta(0) = \varphi_\delta(\beta[n]) < \gamma < \beta = \varphi_\delta(\beta)$. In this counterexample both γ and β are fixed points of φ_δ.

The proofs of (A) and (B) are by simultaneous induction on the pair (γ, β). Notice that if γ is not in the range of φ_δ then (B) is immediate. In fact, either $\gamma \leqslant \varphi_\delta(\beta[n])$ or $\varphi_\delta(\beta[n]) < \gamma$ and we apply Lemma 4.7.3 to $\varphi_\delta(\beta[n])$ and γ: in this way we get $\varphi_\delta(\beta[n]) \leqslant \gamma[1] \leqslant \gamma[n]$ for all $n \geqslant 1$. Therefore we prove (B) only for δ such that γ is in the range, but not a fixed point, of φ_δ.

If $\gamma \leqslant 1$ both (A) and (B) hold because two distinct ordinals below γ do not exist.

If γ is of the form $\omega^{\gamma_0} + \gamma_1$ with $\omega^{\gamma_0} \ggg \gamma_1 > 0$ recall that $\gamma[n] = \omega^{\gamma_0} + \gamma_1[n]$. For (A) assume towards a contradiction that $\gamma[n] < \beta[n] < \gamma < \beta$ and consider different subcases depending on the form of β.

(i) $\beta = \omega^{\beta_0} + \beta_1$ with $\omega^{\beta_0} \ggg \beta_1 > 0$. In this case $\gamma < \beta$ implies $\gamma_0 \leqslant \beta_0$ and $\beta[n] = \omega^{\beta_0} + \beta_1[n] < \gamma$ implies $\beta_0 \leqslant \gamma_0$, so that $\beta_0 = \gamma_0$. Hence $\gamma_1[n] < \beta_1[n] < \gamma_1 < \beta_1$, contradicting the induction hypothesis of (A).

(ii) $\beta = \omega^{\beta_0} > \beta_0$. In this case $\gamma < \beta$ implies $\gamma_0 < \beta_0$ and $\beta[n] = \omega^{\beta_0[n]} \cdot n < \gamma$ implies $\beta_0[n] \leqslant \gamma_0$.

  - If $\beta_0[n] < \gamma_0$ then $\beta[n] < \omega^{\gamma_0} \leqslant \gamma[n]$ against $\gamma[n] < \beta[n]$.
  - If $\beta_0[n] = \gamma_0$ then $\omega^{\gamma_0} \cdot n < \gamma = \omega^{\gamma_0} + \gamma_1$ and hence $\omega^{\gamma_0} \cdot (n-1) < \gamma_1$ so that we can write $\gamma_1 = \omega^{\gamma_0} \cdot (n-1) + \delta$ for some δ with $\omega^{\gamma_0} \ggg \delta > 0$. Then $\gamma_1[n] = \omega^{\gamma_0} \cdot (n-1) + \delta[n] \geqslant \omega^{\gamma_0} \cdot (n-1)$ and $\gamma[n] \geqslant \omega^{\gamma_0} \cdot n = \beta[n]$. This contradicts again $\gamma[n] < \beta[n]$.

(iii) $\beta = \varphi_{\beta_0}(\beta_1)$ with $\beta_0 > 0$ and $\beta_1 < \beta$ or $\beta = \Gamma_\xi$. In this case $\beta[n]$ is in the range of φ_0 while γ is not: applying Lemma 4.7.3 we obtain $\beta[n] \leqslant \gamma[1]$ contradicting $\gamma[n] < \beta[n]$.

There is no need to prove (B) because γ is in the range of no φ_δ.

If γ is of the form ω^{γ_0} with $0 < \gamma_0 < \gamma$ recall that $\gamma[n] = \omega^{\gamma_0[n]} \cdot n$. To prove (A) assume towards a contradiction that $\gamma[n] < \beta[n] < \gamma < \beta$ and consider different subcases depending on the form of β.

(i) $\beta = \omega^{\beta_0} + \beta_1$ with $\omega^{\beta_0} \ggg \beta_1 > 0$. Then $\gamma < \beta$ implies $\gamma_0 \leqslant \beta_0$ and hence $\gamma \leqslant \omega^{\beta_0} + \beta_1[n] = \beta[n]$.

(ii) $\beta = \omega^{\beta_0}$ with $\beta_0 < \beta$. Then $\gamma < \beta$ implies $\gamma_0 < \beta_0$, $\beta[n] = \omega^{\beta_0[n]} \cdot n < \gamma$ implies $\beta_0[n] < \gamma_0$ and $\gamma[n] < \beta[n]$ implies $\gamma_0[n] < \beta_0[n]$. We thus have $\gamma_0[n] < \beta_0[n] < \gamma_0 < \beta_0$ against the induction hypothesis of (A).

(iii) $\beta = \varphi_1(0) = \epsilon_0$. By Lemma 4.7.1 $\beta[n] = \varphi_0^{n+1}(0) \leqslant \gamma[1] \leqslant \gamma[n]$, which contradicts $\gamma[n] < \beta[n]$.

(iv) $\beta = \varphi_1(\beta_1) = \epsilon_{\beta_1}$ with $0 < \beta_1 < \beta$. To simplify the notation let $\xi = \epsilon_{\beta_1[n]} + 1$, so that $\beta[n] = \varphi_0^{n+1}(\xi)$. We have that $\beta[n] < \gamma$ implies $\varphi_0^n(\xi) < \gamma_0$. If $\gamma_0 < \omega^{\varphi_0^{n-1}(\xi)+1}$ then $\varphi_0^n(\xi)$ is the leading term of the Cantor normal form of γ_0 (which includes other terms) and $\gamma_0[n] \geqslant \varphi_0^n(\xi)$. Thus $\gamma[n] > \omega^{\gamma_0[n]} \geqslant \varphi_0^{n+1}(\xi) = \beta[n]$, a contradiction. We can therefore assume $\gamma_0 \geqslant \omega^{\varphi_0^{n-1}(\xi)+1}$. Let γ_1 be such that $\omega^{\gamma_1} \leqslant \gamma_0 < \omega^{\gamma_1+1}$ which, by the above assumption, implies $\gamma_1 > \varphi_0^{n-1}(\xi)$. Consider first the case $\omega^{\gamma_1} < \gamma_0$: in this case the Cantor normal form of γ_0 includes other terms after the leading term ω^{γ_1} and hence $\gamma_0[n] \geqslant \omega^{\gamma_1} > \varphi_0^n(\xi)$ so that $\gamma[n] > \omega^{\gamma_0[n]} > \varphi_0^{n+1}(\xi) = \beta[n]$, a contradiction. We thus have $\gamma_0 = \omega^{\gamma_1}$ and $\gamma = \varphi_0^2(\gamma_1)$. We can repeat this argument until we obtain that $\gamma = \varphi_0^{n+1}(\gamma_n)$ for some $\gamma_n > \xi$. But then $\varphi_0^{n+1}(\gamma_n[n]) < \gamma[n] < \beta[n] = \varphi_0^{n+1}(\xi) < \gamma = \varphi_0^{n+1}(\gamma_n)$ which implies $\gamma_n[n] < \xi < \gamma_n$. Therefore we have $\gamma_n[n] \leqslant \epsilon_{\beta_1[n]}$. Towards a contradiction suppose $\gamma_n[n] = \epsilon_{\beta_1[n]}$. If γ_n is decomposable then it must be $\gamma_n = \epsilon_{\beta_1[n]} + 1$ which cannot be. Otherwise, since $\gamma_n \notin \operatorname{ran}\varphi_1$, it must be $\gamma_n = \omega^\delta > \delta$. In this case $n > 1$ implies that $\gamma_n[n]$ is decomposable and so it cannot be $\epsilon_{\beta_1[n]}$. It follows that $\gamma_n[n] < \epsilon_{\beta_1[n]} < \gamma_n$. Moreover $\gamma_n < \gamma < \beta = \epsilon_{\beta_1}$ and, since γ_n is not a fixed point of φ_1, we contradict the induction hypothesis of (B) with $\gamma_n[n] < \epsilon_{\beta_1[n]} < \gamma_n < \epsilon_{\beta_1}$.

(v) $\beta = \varphi_{\beta_0}(\beta_1)$ with $\beta_0 > 1$ and $\beta_1 < \beta$ or $\beta = \Gamma_\xi$. In this case both $\beta[n]$ and β are fixed points of φ_0 and $\omega^{\gamma_0[n]} \cdot n < \beta[n] < \omega^{\gamma_0} < \beta$ implies $\gamma_0[n] < \beta[n] < \gamma_0 < \beta$ against the induction hypothesis of (A).

We need to prove (B) only for $\delta = 0$ since $\gamma \notin \operatorname{ran}\varphi_1$. If $\omega^{\gamma_0[n]} \cdot n < \omega^{\beta[n]} < \omega^{\gamma_0} < \omega^\beta$ for some β, we have that $\gamma_0[n] < \beta[n] < \gamma_0 < \beta$ against the induction hypothesis of (A).

If γ is $\varphi_1(0) = \epsilon_0$ then $\beta > \epsilon_0$ and Lemma 4.7.2 imply that $\beta[n] \geqslant \epsilon_0$ for all $n \geqslant 1$. Therefore we cannot have a counterexample to (A). On the other hand (B) is immediate because we should consider only the case $\delta = 1$, and $\varphi_1(\beta[n]) < \epsilon_0$ is impossible.

If γ is of the form $\varphi_{\gamma_0}(0)$ with $1 < \gamma_0 < \gamma$ recall that $\gamma[n] = \varphi^{n+1}_{\gamma_0[n]}(0)$. To prove (A) we assume towards a contradiction that $\gamma[n] < \beta[n] < \gamma < \beta$ and consider different subcases depending on the form of β.

(i) $\beta = \omega^{\beta_0} + \beta_1$ with $\omega^{\beta_0} \ggg \beta_1 > 0$. Then, since γ is an indecomposable ordinal, $\gamma < \beta$ implies $\gamma \leqslant \omega^{\beta_0}$ and hence $\gamma \leqslant \omega^{\beta_0} + \beta_1[n] = \beta[n]$.

(ii) $\beta = \varphi_{\beta_0}(0)$ with $\beta_0 < \beta$. Case $\beta_0 = 0$ is trivial. Otherwise $\gamma_0 < \beta_0$ and we have $\varphi^{n+1}_{\gamma_0[n]}(0) < \varphi^{n+1}_{\beta_0[n]}(0) < \varphi_{\gamma_0}(0) < \varphi_{\beta_0}(0)$. Then $\gamma_0[n] < \beta_0[n] < \gamma_0 < \beta_0$, contradicting the induction hypothesis of (A).

(iii) $\beta = \varphi_{\beta_0}(\beta_1)$ with $0 < \beta_1 < \beta$. Notice that $\gamma_0 < \beta_0$ cannot hold, because otherwise $\varphi_{\gamma_0}(\varphi_{\beta_0}(\beta_1[n])) = \varphi_{\beta_0}(\beta_1[n]) < \beta[n] < \gamma$ (the first equality holds because $\beta_0 > \gamma_0$) which is impossible as γ is the least element in the range of φ_{γ_0}. Analogously, $\beta_0 = \gamma_0$ yields $\varphi_{\gamma_0}(\beta_1[n]) = \varphi_{\beta_0}(\beta_1[n]) < \beta[n] < \gamma$ which is impossible for the same reason. Thus $\beta_0 < \gamma_0$ and β does not belong to the range of φ_{γ_0}: Lemma 4.7.3 then implies $\gamma \leqslant \beta[n]$, a contradiction.

(iv) $\beta = \Gamma_0$. Then we are assuming $\varphi^{n+1}_{\gamma_0[n]}(0) < \varphi_{\Gamma_0[n-1]}(0) < \varphi_{\gamma_0}(0) < \Gamma_0$ which implies $\gamma_0[n] < \Gamma_0[n-1] < \gamma_0 < \Gamma_0$. If $n > 2$ (so that $n - 1 > 1$) we have $\gamma_0[n-1] < \Gamma_0[n-1] < \gamma_0 < \Gamma_0$ contradicting the induction hypothesis of (A). In the case $n = 2$, since $\Gamma_0[1] = \epsilon_0$, we get a contradiction with Lemma 4.7.2.

(v) $\beta = \Gamma_\xi$. Let $\delta_{n+1} = \Gamma_{\xi[n]} + 1$ and $\delta_i = \varphi_{\delta_{i+1}}(0)$ for $i \leqslant n$. Thus $\Gamma_\xi[n] = \varphi_{\delta_0}(0)$. Then our assumption $\gamma[n] < \beta[n] < \gamma < \beta$ yields $\gamma_0[n] < \delta_0 < \gamma_0 < \beta$. We claim that $\gamma_0 = \varphi_{\gamma_1}(0)$ for some γ_1. In fact, if γ_0 were decomposable then, since δ_0 is indecomposable, $\gamma_0[n] \geqslant \delta_0$, a contradiction. If instead $\gamma_0 = \varphi_{\gamma_1}(\gamma_2) > \gamma_2 > 0$ then, since $\delta_0 = \varphi_{\delta_1}(0)$, Lemma 4.7.4 implies $\gamma_0[n] \geqslant \delta_0$. This proves the claim. Notice that it must be $\gamma_0 > \gamma_1 > 0$. The claim and the chain of inequalities $\gamma_0[n] < \delta_0 < \gamma_0 < \beta$ yields $\gamma_1[n] < \delta_1 < \gamma_1 < \beta$. Iterating $n + 1$ times we obtain for each $i \leqslant n + 1$ an ordinal γ_i such that $\gamma_{i-1} = \varphi_{\gamma_i}(0)$ and $\gamma_i[n] < \delta_i < \gamma_i < \beta$. In particular $\gamma_{n+1}[n] < \delta_{n+1} < \gamma_{n+1} < \beta$. Since γ_{n+1} is not a Γ ordinal and $\gamma_{n+1} > \delta_0 = \Gamma_{\xi[n]} + 1$, Lemma 4.7.3 implies $\gamma_{n+1}[n] \geqslant \Gamma_{\xi[n]}$. Since $\Gamma_{\xi[n]} + 1 > \gamma_{n+1}[n]$ it must be $\gamma_{n+1}[n] = \Gamma_{\xi[n]}$. Lemma 4.7.5 implies that $\gamma_{n+1} = \Gamma_{\xi[n]} + 1$, contradicting $\Gamma_{\xi[n]} + 1 = \delta_{n+1} < \gamma_{n+1}$.

To prove (B) we should consider only the case $\delta = \gamma_0$: $\varphi_{\gamma_0}(\beta[n]) < \varphi_{\gamma_0}(0)$ is impossible.

If γ is of the form $\varphi_{\gamma_0}(\gamma_1)$ with $0 < \gamma_0$ and $0 < \gamma_1 < \gamma$ we first need to recall $\gamma[n] = \varphi^{n+1}_{\gamma_0[n]}(\varphi_{\gamma_0}(\gamma_1[n]) + 1)$. To prove (A) we assume towards a contradiction that $\gamma[n] < \beta[n] < \gamma < \beta$ and consider different subcases depending on the form of β.

(i) $\beta = \omega^{\beta_0} + \beta_1$ with $\omega^{\beta_0} \gg \beta_1 > 0$. Then, since γ is an indecomposable ordinal, $\gamma < \beta$ implies $\gamma \leqslant \omega^{\beta_0}$ and hence $\gamma \leqslant \omega^{\beta_0} + \beta_1[n] = \beta[n]$.

(ii) $\beta = \varphi_{\beta_0}(0)$ with $\beta_0 < \beta$. Case $\beta_0 = 0$ is trivial. So assume $\beta_0 > 0$, recall that $\beta[n] = \varphi^{n+1}_{\beta_0[n]}(0)$ and notice that $\gamma < \beta$ implies $\gamma_0 < \beta_0$. We consider three different subcases.

- If $\beta_0[n] < \gamma_0$ then we also have $\gamma_0[n] < \beta_0[n]$, against the induction hypothesis of (A). In fact $\beta_0[n] \leqslant \gamma_0[n]$ implies $\beta[n] = \varphi^{n+1}_{\beta_0[n]}(0) \leqslant \varphi^{n+1}_{\gamma_0[n]}(0) < \gamma[n]$, a contradiction.

- If $\beta_0[n] = \gamma_0$ then, peeling off an application of φ_{γ_0} we obtain $\varphi^{n}_{\beta_0[n]}(0) < \gamma_1$. If γ_1 does not belong to the range of $\varphi_{\beta_0[n]} = \varphi_{\gamma_0}$ then Lemma 4.7.3 implies $\varphi^{n}_{\beta_0[n]}(0) \leqslant \gamma_1[n]$ and hence $\beta[n] \leqslant \varphi_{\gamma_0}(\gamma_1[n]) < \gamma[n]$ which is a contradiction. Therefore $\gamma_1 = \varphi_{\gamma_0}(\gamma_2)$ for some $\gamma_2 < \gamma_1$ and we have $\varphi^{n-1}_{\beta_0[n]}(0) < \gamma_2$. If γ_2 does not belong to the range of $\varphi_{\beta_0[n]}$ then Lemma 4.7.3 implies $\varphi^{n-1}_{\beta_0[n]}(0) \leqslant \gamma_2[n]$ and consequently $\beta[n] \leqslant \varphi^{2}_{\gamma_0}(\gamma_2[n]) < \varphi_{\gamma_0}(\varphi_{\gamma_0}(\gamma_2[n]) + 1) < \varphi_{\gamma_0}(\gamma_1[n]) < \gamma[n]$. Therefore $\gamma_2 = \varphi_{\gamma_0}(\gamma_3)$ for some $\gamma_3 < \gamma_2$ and we iterate this procedure finding for each $i \leqslant n+1$ some γ_i such that $\varphi_{\gamma_0}(\gamma_i) = \gamma_{i-1}$ and $\varphi^{n-i+1}_{\beta_0[n]}(0) < \gamma_i$. Then, starting from $0 \leqslant \gamma_{n+1}[n]$ we obtain $\beta[n] \leqslant \varphi^{n+1}_{\gamma_0}(\gamma_{n+1}[n]) < \cdots < \varphi^{i}_{\gamma_0}(\gamma_i[n]) < \cdots < \varphi_{\gamma_0}(\gamma_1[n]) < \gamma[n]$, which is again a contradiction.

- If $\gamma_0 < \beta_0[n]$ then γ is not in the range of $\varphi_{\beta_0[n]}$ and Lemma 4.7.3 implies that $\beta[n] \leqslant \gamma[n]$.

(iii) $\beta = \varphi_{\beta_0}(\beta_1)$ with $0 < \beta_1 < \beta$. As before, we consider different subcases.

- If $\beta_0 < \gamma_0$ then β is not in the range of φ_{γ_0} and Lemma 4.7.3 implies that $\gamma \leqslant \beta[n]$.

- If $\beta_0 = \gamma_0$ then we have $\gamma_1 < \beta_1$ and, since $\beta_0[n] = \gamma_0[n]$, also $\gamma_1[n] < \beta_1[n]$. The induction hypothesis of (A) then yields that $\gamma_1 \leqslant \beta_1[n]$ and hence $\gamma \leqslant \varphi_{\gamma_0}(\beta_1[n]) < \varphi_{\beta_0}(\beta_1[n]) + 1 < \beta[n]$ a contradiction.

- If $\beta_0[n] < \gamma_0 < \beta_0$ then by the induction hypothesis of (A) we have $\beta_0[n] \leqslant \gamma_0[n]$. Since $\varphi_{\gamma_0}(\varphi_{\beta_0}(\beta_1[n])) = \varphi_{\beta_0}(\beta_1[n]) < \beta[n] < \gamma$ (where the equality holds because $\gamma_0 < \beta_0$) we have, peeling off an application of φ_{γ_0}, that $\varphi_{\beta_0}(\beta_1[n]) < \gamma_1$. Notice also that $\gamma_1 < \gamma < \beta$ so that we get $\varphi_{\beta_0}(\beta_1[n]) < \gamma_1 < \varphi_{\beta_0}(\beta_1)$. We know that γ_1 is not a fixed point of φ_{γ_0} and, a fortiori, of φ_{β_0}. Hence, by the induction hypothesis of (B), $\varphi_{\beta_0}(\beta_1[n]) \leqslant \gamma_1[n]$. Applying φ_{γ_0} to both members of the previous inequality, we get $\varphi_{\beta_0}(\beta_1[n]) + 1 \leqslant \varphi_{\gamma_0}(\gamma_1[n]) + 1$. Finally, applying

$n+1$ times $\varphi_{\beta_0[n]}$ to the left hand side and $\varphi_{\gamma_0[n]}$ to the right hand side of the previous inequality (in addition to the fact that $\beta_0[n] \leqslant \gamma_0[n]$), we get precisely $\beta[n] \leqslant \gamma[n]$.

- If $\beta_0[n] = \gamma_0$ then to ease the notation let $\xi = \varphi_{\beta_0}(\beta_1[n]) + 1$ so that $\beta[n] = \varphi^{n+1}_{\beta_0[n]}(\xi)$. Then we have $\varphi^{n}_{\beta_0[n]}(\xi) < \gamma_1$ by peeling off one application of φ_{γ_0} from both sides. If γ_1 does not belong to the range of $\varphi_{\beta_0[n]} = \varphi_{\gamma_0}$ then Lemma 4.7.3 implies $\varphi^{n}_{\beta_0[n]}(\xi) \leqslant \gamma_1[n]$ and hence $\beta[n] \leqslant \varphi_{\gamma_0}(\gamma_1[n]) < \gamma[n]$ which is a contradiction. Therefore $\gamma_1 = \varphi_{\gamma_0}(\gamma_2)$ for some $\gamma_2 < \gamma_1$ and we have $\varphi^{n-1}_{\beta_0[n]}(\xi) < \gamma_2$ again by peeling off one application of φ_{γ_0} from both sides. If γ_2 does not belong to the range of $\varphi_{\beta_0[n]}$ then Lemma 4.7.3 implies $\varphi^{n-1}_{\beta_0[n]}(\xi) \leqslant \gamma_2[n]$ and hence $\beta[n] \leqslant \varphi^2_{\gamma_0}(\gamma_2[n]) < \varphi_{\gamma_0}(\gamma_1[n]) < \gamma[n]$. Therefore $\gamma_2 = \varphi_{\gamma_0}(\gamma_3)$ for some $\gamma_3 < \gamma_2$ and we iterate this procedure finding for each $i \leqslant n+1$ some γ_i such that $\varphi_{\gamma_0}(\gamma_i) = \gamma_{i-1}$ and $\varphi^{n-i+1}_{\beta_0[n]}(\xi) < \gamma_i$. Then we have $\varphi_{\beta_0}(\beta_1[n]) < \xi < \gamma_{n+1}$ and γ_{n+1} is not a fixed point for φ_{γ_0} and hence not in the image of φ_{β_0} because $\beta_0 > \beta_0[n] = \gamma_0$. By Lemma 4.7.3 $\varphi_{\beta_0}(\beta_1[n]) \leqslant \gamma_{n+1}[1]$. Notice that if γ_{n+1} is a successor then $\varphi_{\beta_0}(\beta_1[n]) < \xi \leqslant \gamma_{n+1}[1]$ and if γ_{n+1} is limit then $\gamma_{n+1}[1] < \gamma_{n+1}[n]$. Therefore $\varphi_{\beta_0}(\beta_1[n]) < \gamma_{n+1}[n]$ and hence $\xi \leqslant \gamma_{n+1}[n]$. Thus, applying $n+1$ times φ_{γ_0} to both sides of this inequality, we get $\beta[n] \leqslant \varphi^{n+1}_{\gamma_0}(\gamma_{n+1}[n]) < \ldots < \varphi^{i}_{\gamma_0}(\gamma_i[n]) < \ldots < \varphi_{\gamma_0}(\gamma_1[n]) < \gamma[n]$.
- If $\gamma_0 < \beta_0[n]$ then γ is not in the range of $\varphi_{\beta_0[n]}$ and Lemma 4.7.3 implies that $\beta[n] \leqslant \gamma[n]$.

(iv) $\beta = \Gamma_0$. We are assuming $\varphi^{n+1}_{\gamma_0[n]}(\varphi_{\gamma_0}(\gamma_1[n]) + 1) < \varphi_{\Gamma_0[n-1]}(0) < \varphi_{\gamma_0}(\gamma_1) < \Gamma_0$ and consider three subcases.

- If $\Gamma_0[n-1] > \gamma_0$ then $\gamma_1 > \Gamma_0[n]$. Since $\gamma_1[n] < \gamma[n]$ we would get $\gamma_1[n] < \Gamma_0[n] < \gamma_1 < \Gamma_0$, contradicting the inductive hypothesis of (A).
- If $\Gamma_0[n-1] = \gamma_0$ then $\Gamma_0[n] < \varphi_{\gamma_0}(\gamma_1[n]) + 1 < \gamma[n]$, a contradiction.
- Otherwise $\gamma_0[n] < \Gamma_0[n-1] < \gamma_0 < \Gamma_0$ and we can argue as in (iv) of case $\gamma = \varphi_{\gamma_0}(0)$.

(v) $\beta = \Gamma_\xi$. Let δ be such that $\beta[n] = \varphi_{\varphi_\delta(0)}(0)$ and notice that $\varphi_\delta(0) > \gamma_0[n]$. With this notation, we are assuming $\varphi^{n+1}_{\gamma_0[n]}(\varphi_{\gamma_0}(\gamma_1[n]) + 1) < \varphi_{\varphi_\delta(0)}(0) < \varphi_{\gamma_0}(\gamma_1) < \Gamma_\xi$ and consider different subcases.

- If $\gamma_0 > \varphi_\delta(0)$. Thus $\gamma_0[n] < \varphi_\delta(0) < \gamma_0 < \beta$ and so $\varphi_{\gamma_0[n]}(0) < \varphi_{\varphi_\delta(0)}(0) = \beta[n] < \varphi_{\gamma_0}(0) < \varphi_\beta(0) = \beta$. Since $\varphi_\delta(0) > \gamma_0[n]$ then $\beta[n]$ is a fixed point of $\varphi_{\gamma_0[n]}$ and we get $\varphi^{n+1}_{\gamma_0[n]}(0) < \beta[n] < \varphi_{\gamma_0}(0) < \beta$. This contradicts the induction hypothesis of (A).

- If $\gamma_0 = \varphi_\delta(0)$ then $\varphi^{n+1}_{\gamma_0[n]}(0) < \gamma[n] < \beta[n] = \varphi_{\gamma_0}(0) < \gamma$ contradicting the inductive hypothesis of (A).
- If $\gamma_0 < \varphi_\delta(0)$ then both β and $\beta[n]$ are fixed points of φ_{γ_0}. Hence peeling off one application of φ_{γ_0} we get $\beta[n] < \gamma_1 < \beta$. Moreover, $\gamma_1[n] < \gamma[n] < \beta[n]$ contradicts the inductive hypothesis of (A).

To prove (B) we should consider only the case $\delta = \gamma_0$ and assume towards a contradiction that $\gamma[n] < \varphi_{\gamma_0}(\beta[n]) < \gamma < \varphi_{\gamma_0}(\beta)$. Then $\beta[n] < \gamma_1 < \beta$ and the induction hypothesis of (A) implies $\beta[n] \leqslant \gamma_1[n]$ so that $\varphi_{\gamma_0}(\beta[n]) \leqslant \varphi_{\gamma_0}(\gamma_1[n]) < \gamma[n]$, a contradiction.

If γ is of the form Γ_0 then $\beta > \Gamma_0$ and Lemma 4.7.2 imply that $\beta[n] \geqslant \Gamma_0$ for all $n \geqslant 1$. Therefore we cannot have a counterexample to (A). On the other hand (B) is immediate because we should consider only the case $\delta = \Gamma_0$, and $\varphi_{\Gamma_0}(\beta[n]) < \Gamma_0 = \varphi_{\Gamma_0}(0)$ is impossible.

If γ is of the form Γ_ξ then to prove (A) assume towards a contradiction that $\gamma[n] < \beta[n] < \gamma < \beta$ and consider different subcases depending on the form of β.

(i) $\beta \neq \Gamma_\delta$ for all δ. In this case $\beta > \Gamma_\xi$ and Lemma 4.7.3 imply that $\gamma \leqslant \beta[n]$ a contradiction.

(ii) $\beta = \Gamma_\delta$ with $\delta < \beta$. In this case $\xi < \delta$ and $\Gamma_\xi[n] < \Gamma_\delta[n] < \Gamma_\xi < \Gamma_\delta$. By considering $n+1$ times the index of the Veblen functions, the first inequality becomes $\Gamma_{\xi\lceil n\rceil} < \Gamma_{\delta\lceil n\rceil}$. Then we get $\xi\lceil n\rceil < \delta\lceil n\rceil < \xi < \delta$ contradicting the smoothness of the system of fundamental sequences on ζ we started from.

To prove (B) we should only consider the case of φ_{Γ_ξ}. This is immediate since $\varphi_{\Gamma_\xi}(\beta[n]) < \Gamma_\xi = \varphi_{\Gamma_\xi}(0)$ is impossible. □

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